Tensor decompositions: invariance and computational complexity

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Abstract

Tensors are combinations of several vectors such that a bigger vector space, also called the tensor space, emerges. The tensor space has a richer structure than that of the separate vector spaces. Tensors are universally used in quantum mechanics to model multipartite systems. Tensor decompositions aim to write these tensors as combinations of elements of the original vector spaces, and have been used historically to ease computations and to gain insight in the properties of tensors. More recently, these decompositions have been used to study pure states with high success, but mixed states have presented more challenges due to their positivity structure. In this work we present a general framework to encode physical structure and symmetries for tensor decompositions. The main result shows that every tensor that fulfills the symmetry conditions imposed by the physical structure can be finitely decomposed in this framework. To achieve this we introduce three separate decompositions, two of which explicitly exhibit convex properties relevant to physics: positivity and separability. We study these decompositions, both separately and their relations. We also relate particular cases of these decompositions to previously studied decompositions of nonnegative matrices, showing that this is a generalisation. Moreover, we study the computational complexity of related problems. We review complexity classes that have been useful to solve related problems and how these classes can relate to our problems.
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1. Introduction

There are plenty of examples as to how systems with a large number of components behave differently to systems with a small number of the same components, such as the weather and condensed matter physics. Therefore we need tools to study these systems accurately, in particular an efficient description of its elements. In this work we focus on how a growing number of vector spaces in the definition of the tensor space leads to elements that are not efficiently representable, and how to efficiently express relevant corners of tensor space.

Take for example the rank of a matrix. Let $M$ be a complex square matrix of dimension $d$ and rank $r$. Then

$$M = PQ$$

with $P$, $Q$ of dimension $d \times r$ (or $r \times d$). If we count the number of complex entries in each representation, $M$ needs $d^2$ entries, while $P$ and $Q$ need $dr$ each, for a total of $2dr$. The rank is bounded by $d$, so in the worst case there are twice as many entries. But what if $r$ is significantly smaller than $d$? In this case, we obtain a much more efficient representation.

This is particularly true in multipartite quantum systems. Assume we have $n$ $d$-dimensional quantum systems. Since the matrix that describes the whole state is part of a tensor product, it has $O(d^n)$ real entries. This grows exponentially with the number of systems, which makes the state intractable for large values of $n$. For example, let $n = 275$ with $d = 2$, then

$$d^n = 2^{275} \sim 10^{82},$$

which is the estimated number of atoms in the universe. Note that 275 is very small if we consider, for example, the number of particles in a typical condensed matter problem. Meanwhile, a tensor decomposition of rank $r$, would contain at most $\sim O(ndr)$ real entries, $\sim O(dr)$ if the state is symmetric. As before, this offers no advantage for large ranks, but for small ranks it turns into a very efficient representation.

This has been done for pure states with high success [20], but a good characterisation for general quantum states is still lacking [7], see Section 2.3. Its study would be useful for condensed matter physics, which studies large many-body systems, as well as for quantum computing with large number of qubits, to name two examples.

It is well known [11] that an element of a tensor product of vector spaces, $v \in V_0 \otimes \cdots \otimes V_n$, can be decomposed as a finite sum of elementary tensors

$$a^{[0]} \otimes \cdots \otimes a^{[n]},$$
where $a^{[i]} \in \mathcal{V}_i$ for all $i$. The indices in the sum can be arranged in different ways [15] and, in physics, this arrangements are used to reflect the physical structure of the system. Two prominent examples are nearest neighbour interactions on a circle

\[ v = \sum_{\alpha_0, \ldots, \alpha_n = 1}^r a^{[0]}_{\alpha_0} \otimes a^{[1]}_{\alpha_1} \otimes a^{[n-1]}_{\alpha_{n-1}} \otimes a^{[n]}_{\alpha_n}, \]

where two adjacent vector spaces share one of the induces $\alpha_i$, and a single index

\[ v = \sum_{\alpha = 1}^r a^{[0]}_\alpha \otimes a^{[1]}_\alpha \otimes \cdots \otimes a^{[n]}_\alpha, \]

where every vector space shares a single index $\alpha$ with all others. In both examples, the concept of size of the decomposition is represented by the number of factors in the sum, denoted by $r$. The minimum $r$ possible defines the rank of an element $v$ under a particular tensor network. For these examples, these ranks are called operator Schmidt rank and tensor rank, respectively [6].

Symmetries are very important in physics and mathematics, since they characterise conserved quantities and allow for less degrees of freedom, thus allowing for more efficient parametrisation of systems. In Section 2.1.2 we characterise symmetries through group actions, but we focus in a particular kind of symmetries: external symmetries. An external symmetry is a permutation of the subsystems that conform a multipartite physical system. For example, consider a separable state in a bipartite system

\[ \rho_1 \otimes \rho_2. \]

The only nontrivial global symmetry this state can have is parity, that is that

\[ \rho_1 \otimes \rho_2 = P \rho_1 \otimes \rho_2 P^\dagger = \rho_2 \otimes \rho_1, \]

where $P$ acts as $P |\psi_1, \psi_2 \rangle = |\psi_2, \psi_1 \rangle$. In contrast, there are internal symmetries which are symmetries of each individual subsystem. In the same example, consider a matrix representation of some group $G$, with $g \mapsto U_g$. Then internal symmetries are of the form

\[ (U_g \otimes I) \rho_1 \otimes \rho_2 (U_g^\dagger \otimes I) = \rho_1 \otimes \rho_2. \]

A global symmetry of nearest neighbour interactions on a circle is the translational symmetry. An invariant decomposition in this case would be

\[ v = \sum_{\alpha_0, \ldots, \alpha_n = 1}^r a_{\alpha_0} a_{\alpha_1} \otimes a_{\alpha_1} a_2 \otimes \cdots \otimes a_{\alpha_{n-1}} a_n \otimes a_{\alpha_n}. \]

The tensor decomposition with a single index admits any permutation subgroup of the full permutation group $S_n$ as a symmetry. If we consider the full permutation group, then the decomposition is

\[ v = \sum_{\alpha = 1}^r a_{\alpha} \otimes a_{\alpha} \otimes \cdots \otimes a_{\alpha}. \]
These aforementioned particular cases have been studied before, see [6].

Chapter 2 introduces the weighted simplicial complex with group action, the object we use to encode the physical properties of tensors, as well as the definition and results regarding tensor decompositions over these weighted simplicial complexes. In Chapter 3 we introduce decision problems associated to the tensor decompositions and discuss preliminary results regarding the complexity of these problems. Finally, Chapter 4 contains a summary as well as further work on the subject.
2. Tensor factorisation over weighted simplicial complexes

In this chapter we describe a new framework for tensor decomposition. We use weighted simplicial complexes to encode the physical structure and groups for symmetries. Intuitively, the structure of our tensor can be seen as a tensor product of elements living on vertices of a weighted simplicial complex that obey certain symmetry conditions imposed by the group action. We define an invariant (with respect to the group) decomposition as well as invariant separable decomposition and invariant purification [6]. The main result regards the existence of these decompositions. We show relations between each type of decomposition and, finally, we apply these results to decomposition of nonnegative tensors, a generalisation of current decompositions of nonnegative matrices [9, 17]. This work can be found in [5].

In Section 2.1 we introduce the underlying structure we use as well as the symmetry characterisation and examples on specific physical systems; in Section 2.2 we define the invariant tensor decompositions and show the existence theorem, in Section 2.3 we compare the different ranks and finally in Section 2.4 we show how these decompositions can be applied to study decompositions of nonnegative tensors.

2.1 Weighted simplicial complexes and group actions

In this section we introduce the concepts of weighted simplicial complex and group action, which provide the underlying structure we use for tensor decompositions. We consider weighted simplicial complexes instead of simplicial complexes because of Theorem 2.2, in combination with Proposition 2.4.

2.1.1 Weighted simplicial complexes

To define weighted simplicial complexes we first introduce simplexes and simplicial complexes in terms of the definition we use, which sees these objects as functions).

Let $n \in \mathbb{N}$, we denote by $[n]$ the set $\{0, 1, \ldots, n\}$ and by $\mathcal{P}_n$ the power set\(^1\) of $[n]$, $\mathcal{P}([n])$. Moreover, denote by $\mathcal{P}_\mathbb{N}$ the power set of $\mathbb{N}$, which can be defined as

$$
\mathcal{P}_\mathbb{N} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n.
$$

\(^1\)The power set is the set of subsets of a set. I.e. $\mathcal{P}(A) = \{A' \subseteq A\}$. For example, for $A = \{a, b\}$, $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, A\}$. 13
Let us first define a simplex, which we first generalise to a simplicial complex and finally to weighted simplicial complex. A $k$-simplex is the convex hull of $k+1$ affine independent points, which are called vertices. A vertex is also a 0-simplex. Fig. 2.1 shows graphical examples of low dimensional simplexes. A simplex is a closed object.

\begin{figure}
\centering
\includegraphics[width=\textwidth]{simplexes.png}
\caption{From left to right, 1-simplex, 2-simplex and 3-simplex.}
\end{figure}

Therefore a $k$-simplex contains $\binom{k+1}{s+1}$ $s$-simplexes $\forall s \leq k$, because the convex hull of $k+1$ points includes the convex hull of $k+1$ combinations of $k$ points, $k(k+1)/2$ combinations of $k-1$ points, and so on. Since we are only interested in the topological properties of simplexes, regardless of the geometry of the space they form, $k$-simplex can be defined as the characteristic function of the set $[k] \subseteq \mathbb{N}$, closed by taking subsets, in the following way:

**Definition 2.1 (Simplex).** A $k$-simplex is a function

$$\omega : \mathcal{P}_n \rightarrow \{0, 1\}$$

such that $\omega(S) = 1$ if and only if $S \subseteq [k]$.

This definition allows us to see $k$-simplexes as functions of a broader power set, $\mathcal{P}_n$. Since for all elements $S$ of $\mathcal{P}_n$ that contain an element not in $[k]$, $\omega(S) = 0$, we can consider a restricted version of $\omega$ to $[n]$, for $n \geq k$. Then a $k$-simplex is the characteristic function of $[k] \subseteq \mathcal{P}_n$ closed by passing subsets. All $n \geq k$, as well as $\mathbb{N}$ yield the same structure. Definition 2.1 allows us to naturally define simplicial complexes:

**Definition 2.2 (Simplicial complex).** A simplicial complex on $[n]$ is a function

$$\Omega : \mathcal{P}_n \rightarrow \{0, 1\}$$

such that if $S' \subseteq S$ and $\Omega(S) = 1$, then $\Omega(S') = 1$.

A set $S \in \mathcal{P}_n$ is called a face if $\Omega(S) = 1$ and a maximal simplex or facet if $\Omega(S) = 1$ and $\forall S' \subseteq S'$, $\Omega(S') = 0$. The set of facets in a simplicial complex is denoted by $\mathcal{F}$. That is that, given a simplicial complex $\Omega$ over $[n]$ then

$$\mathcal{F} = \{S \in \mathcal{P}_n \text{ such that } \Omega(S) = 1 \text{ and if } S \subseteq S' \text{ then } \Omega(S') = 0\}.$$  

For a vertex $i$, we denote by $\mathcal{F}_i$ the set of facets that contain $i$.  

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Informally, a simplicial complex can be seen as the gluing together of several simplexes through their faces, as pictured in Fig. 2.2. For example, a simplex is a particular case of a simplicial complex; we denote by $\Sigma_n$ the $n$-simplex.

We need an additional generalisation on simplicial complexes: weighted simplicial complexes [4].

**Definition 2.3 (Weighted simplicial complex).** A weighted simplicial complex (wsc) on $[n]$ is a function

$$\Omega : \mathcal{P}_n \rightarrow \mathbb{N}$$

such that

$$S' \subseteq S \Rightarrow \Omega(S') | \Omega(S).$$

This definition gives weights to the faces. We define the set of facets of a wsc $\Omega$ as

$$\mathcal{F} = \{S \in \mathcal{P}_n \text{ such that } \Omega(S) \neq 0 \text{ and if } S \subseteq S' \text{ then } \Omega(S') = 0\}. $$

We denote by $\mathcal{F}_i$ the set of facets that contain a vertex $i$. $\mathcal{F}$ does not take into account the weighted structure of $\Omega$. Let $F \in \mathcal{F}$; then consider the set $\{(F,1), \ldots, (F,\Omega(F))\} \subset \mathcal{P}_n \times \mathbb{N}$. Define the set of weighted facets $\tilde{\mathcal{F}}$ as

$$\tilde{\mathcal{F}} = \bigcup_{F \in \mathcal{F}}\{(F,0), \ldots, (F,\Omega(F) - 1)\}.$$ 

This definition ensures that each facet is contained in $\tilde{\mathcal{F}}$ $\Omega(F)$ times. We call the elements $(F,m) \in \tilde{\mathcal{F}}$ weighted facets or, in short, weights\(^2\) of the facet $F$. We denote $\forall i \in [n]$ by $\tilde{\mathcal{F}}_i$ the restriction of $\tilde{\mathcal{F}}$ to elements whose facet contains $i$. We can define a collapse map $c$ that is nothing but the projection of $\tilde{\mathcal{F}}$ to the first component:

$$c : \tilde{\mathcal{F}} \rightarrow \mathcal{F}$$

$$(F,l) \mapsto F$$

\(^2\)The weights of $F$ are elements of $\tilde{\mathcal{F}}$, while the weight of $F$ is $\Omega(F)$. Therefore a weight of $F$ is an element of $\tilde{\mathcal{F}}$, but the weight of $F$ is a natural number.
Two vertices $i, j$ in a wsc $\Omega$ are neighbours if $F_i \cap F_j \neq \emptyset$. Two vertices $i, j$ are connected if there exists a sequence of vertices $\{i_\alpha\}_{\alpha=1}^k$ such that $i = i_1$, $i_{l-1}$ neighbours $i_l \forall l \leq k$ and $i_k = j$. A wsc is connected if any two vertices are connected.

Figure 2.3: Examples of (connected) wscs. From left to right, simple edge, double edge, triple edge.

**Example 2.1.** *An important subclass of weighted simplicial complexes are graphs (see Fig. 2.4). A connected graph with more than one vertex can be understood as a weighted simplicial complex over its vertices where all facets are 2-simplexes.*

The facets of $\Omega$ are the vertices of the graph, $\{0, 1\}, \{0, 2\}, \{1, 2\}, \{2, 3\}$, with weight 1. Each facet is a 1-simplex. Every other non-singular element $S$ of $\mathcal{P}_4$ fulfills $\Omega(S) = 0$.

Figure 2.4: Example of a graph and its associated wsc $\Omega$.

We finally define a refinement of a given wsc $\Omega$. Intuitively, a refinement of $\Omega$ is another wsc, $\Omega'$, on the same set of vertices $[n]$. $\Omega'$ fulfills that each facet has increased or equal weight, and every element with weight 0 keeps that weight. Fig. 2.3 shows two refinements of the single edge. Formally:

**Definition 2.4.** Let $\Omega$ be a wsc on $[n]$. A refinement of $\Omega$ is another wsc $\Omega'$ on the same $[n]$ such that $\forall S \in \mathcal{P}_n$:

1. $\Omega(S) = 0 \Rightarrow \Omega'(S) = 0$
2. $\Omega(S) \leq \Omega'(S)$.

Note that condition i) states that $\Omega$ and $\Omega'$ have the same facet structure, i.e. the facets are the same for both wscs, and condition ii) that the weights on $\Omega'$ are higher.
2.1.2 Group actions

Let $G$ be a group. A group action of $G$ on a set $X$ describes how $G$ shuffles the elements of $X$. Formally:

**Definition 2.5** (Group action [8]). Let $G$ be a group and $X$ a set. A group action of $G$ on $X$ is a map $G \times X \rightarrow X$, with the image of $(g, x)$ denoted $g \cdot x$, such that

i) $\forall x \in X, e \cdot x = x$.

ii) $(gh) \cdot x = g \cdot (h \cdot x)$ $\forall g, h \in G, x \in X$.

For such an action we say that a group $G$ acts on $X$. Note that we can see each $g \in G$ as an automorphism

$$g : X \rightarrow X$$

$$x \mapsto g \cdot x.$$

We are interested in particular types of actions:

**Definition 2.6.** Let $G$ be a group and $X$ a set. Then

i) The action of a group $G$ on $X$ is free if $\forall x \in X$

$$g \cdot x = x \Rightarrow g = e.$$

ii) The action of a group $G$ on $[n]$ is blending if

$$\{g_0 \cdot 0, \ldots, g_n \cdot n\} = [n] \Rightarrow \exists g \in G \mid g \cdot i = g_i \cdot i \forall i \in [n]$$

**Example 2.2.** Some examples of group actions:

i) Let $G$ be a group and $X$ a set. The trivial action is $g \cdot x = x \forall x \in X, g \in G$.

ii) Let $G$ be a group and set $X = G$. Then

$$G \times G \rightarrow G$$

$$(g, h) \mapsto gh$$

is an action of $G$ on itself.

iii) Let $X = [n]$ and $G = C_n$, the cyclic group of $n + 1$ elements. $C_n$ acts on $[n]$ as

$$c \cdot i = i + 1$$

$\forall i \in [n]$, where $c$ is a generator of the cyclic group. The other group elements’ actions follow from the definition of group action. Note that this action is free.
Let $\Omega$ be a wsc and consider a group action on a set of vertices $[n]$. We want this group action to be compatible with the structure of $\Omega$. First we need to define how group actions are compatible with maps:

**Definition 2.7.** Let $G$ be a group action on $X$. Then:

i) Let $G$ act on a set $Y$. A map $f : X \to Y$ is $G$-linear if it is compatible with the group action. I.e.

$$f(g \cdot x) = g \cdot f(x)$$

$\forall g \in G, x \in X$. If $G$ acts trivially on $Y$ we say that $f$ is $G$-invariant.

ii) From a map $f : X \to Y$ and $g \in G$ define

$$g f : X \to Y \quad x \mapsto f(g^{-1} \cdot x).$$

This fulfills $g(h f) = gh f \forall h \in G$ and $e f = f$. If $f$ is defined on a subset $X' \subseteq X$ then $g f$ has domain $g X' = \{g \cdot x \in X \mid x \in X'\}$.

Moreover, a group action on a set of vertices can be canonically\(^3\) extended to the power set in the following way. Given $\Omega$ a wsc on $[n]$, $G$ group acting on $[n]$ and $S = \{i_0, \ldots, i_l\} \in \mathcal{P}_n$, the action of $G$ on $S$ is

$$g \cdot S = g \cdot \{i_0, \ldots, i_l\} = \{g \cdot i_0, \ldots, g \cdot i_l\} \quad \forall g \in G.$$

**Proposition 2.1.** Let $\Omega$ be a wsc on $[n]$ and let $G$ be a group acting on $[n]$ and on $\mathcal{P}_n$ such that

$$g \cdot S = g \cdot \{i_0, \ldots, i_l\} = \{g \cdot i_0, \ldots, g \cdot i_l\} \quad \forall S \in \mathcal{P}_n, \forall g \in G.$$

If $\Omega$ is $G$-invariant this action can be restricted to a group action on the set of facets $\mathcal{F} \subseteq \mathcal{P}_n$.

**Proof.** We only need to show that the action on $\mathcal{F}$ is closed $\forall g \in G$, since the other properties of the group action come from $\mathcal{F} \subseteq \mathcal{P}_n$. That is

$$\forall g \in G, F \in \mathcal{F} \quad g \cdot F \in \mathcal{F}.$$  

We show the contrapositive statement, that is

$$\exists F \in \mathcal{F}, g \in G \text{ such that } g \cdot F \notin \mathcal{F} \quad \Rightarrow \quad \Omega \text{ is not } G\text{-invariant}.$$  

Let $F \in \mathcal{F}$ and $g \in G$ such that $g \cdot F \notin \mathcal{F}$, i.e. $\exists S \in \mathcal{P}_n$ such that $g \cdot F \subseteq S$ and $\Omega(S) \neq 0$. Consider $g^{-1} \cdot S$, this element fulfills $F \subseteq g^{-1} \cdot S$. Since $g^{-1} \cdot S$ contains $F$, a facet by hypothesis, we have $\Omega(g^{-1} \cdot S) = 0$. Thus

$$\Omega(g^{-1} \cdot S) \neq \Omega(S)$$

and $\Omega$ is not $G$-invariant. \hfill \Box

\(^{3}\text{i.e. in the most natural way.}\)
Definition 2.8 (Group action on a wsc). Let $\Omega$ be a wsc on $[n]$ and let $G$ be a group acting on $[n]$. Then we say $G$ acts on $\Omega$ if

i) $\Omega$ is $G$-invariant.

ii) There exists a group action of $G$ on $\tilde{F}$ such that the collapse map

$$
c : \tilde{F} \to F$$

$$(F, l) \mapsto F$$

is $G$-linear (we will usually identify the action on $\tilde{F}$ with the action on $[n]$, or even identify both of them by the group).

We sometimes denote a group action $g \cdot i = gi$. Note that this is well defined by Proposition 2.1. Finally, we need the definition of a free group action on a wsc.

Definition 2.9. Let $\Omega$ be a wsc on $[n]$ and $G$ a group acting on $\Omega$. We say that $G$ acts freely on $\Omega$ if the action of $G$ on $\tilde{F}$ is free (seeing $\tilde{F}$ as a set and taking freeness as defined in Definition 2.6).

Note that a group can act freely on the set of vertices but not freely on the wsc. Consider the following example:

Example 2.3. Let $\Omega$ be the full simplex on $[n]$, $\Sigma_n$ and $G$ the cyclic group $C_n$, with generator $c$, acting as

$$c^m \cdot i = m + i \pmod{n + 1} \quad \forall i \in [n].$$

This action is free on $[n]$. The only possible action on $\tilde{F}$ is the identity, since $|\tilde{F}| = 1$, which is not free for $n \geq 1$.

Example 2.4. Consider the simplicial complex on $[3] = \{0, 1, 2, 3\}$ in Fig. 2.5 with weighted facets $\tilde{F} = F = \{a, b\}$. Let $G = \{e, g\}$ be the group with 2 elements with the following group action on $[3]$:

$$g \cdot 2 = 2 \quad g \cdot 3 = 3$$

$$g \cdot 1 = 0 \quad g \cdot 0 = 1.$$
Thus $g \cdot a = a$ and $g \cdot b = b$. This action is, therefore, not free on $\Omega$, because it is not free on $\tilde{\mathcal{F}}$. Let us raise the weights of the facets to 2, such that $\tilde{\mathcal{F}}' = \{a_0, a_1, b_0, b_1\}$. This defines a new wsc on $[n]$ that we denote by $\Omega'$. Let $G$ act on $\tilde{\mathcal{F}}'$ as

$$g \cdot a_0 = a_1 \quad g \cdot a_1 = a_0$$

$$g \cdot b_0 = b_1 \quad g \cdot b_1 = b_0.$$ 

\[
\begin{array}{c}
\tilde{\mathcal{F}} 
\downarrow^g
\rightarrow \\
\tilde{\mathcal{F}}' 
\downarrow^g
\rightarrow \\
\mathcal{F}
\end{array}
\]

Figure 2.6: Commutative diagram of $G$-linearity of the collapse map $c$.

The collapse map $c$ is clearly $G$-linear and the action on $\tilde{\mathcal{F}}'$ is free, therefore the action on $\Omega'$ is free. Note that saying that $c$ is $G$-linear is equivalent to the commutative diagram in Fig. 2.6. Note that in the diagram $g$ denotes the element $g$ acting on elements of $\tilde{\mathcal{F}}$ and $\mathcal{F}$, which are two separate actions.

Consider a wsc $\Omega$ on $[n]$ and a group $G$ acting on $\Omega$. Consider the effect of a group element on a single vertex. That is consider

$$g_{ji} : \tilde{\mathcal{F}}_i \rightarrow \tilde{\mathcal{F}}$$

$$(F, m) \mapsto g \cdot (F, m) = (F', m').$$

From $G$-linearity, this commutes with the collapse map, thus $F' = g \cdot F = F'$ $\forall m$. Therefore the weights of a facet $F$ map to the weights of the facet $g \cdot F$. Moreover, clearly $g \cdot F \in \tilde{\mathcal{F}}_{gi}$, so we can write $g_{ji}$ as

$$g_{ji} : \tilde{\mathcal{F}}_i \rightarrow \tilde{\mathcal{F}}_{gi}$$

$$(F, m) \mapsto (g \cdot F, m')$$

This means that the action on $\mathcal{F}$ nicely preserves the structure of $\Omega$, i.e.

$$|F| = |g \cdot F| \quad \forall F \in \mathcal{F}, g \in G$$

and

$$g \tilde{\mathcal{F}}_i = \tilde{\mathcal{F}}_{gi} \quad \forall i \in [n], g \in G.$$ 

Thus a group action of $G$ on a set of nonweighted facets $\mathcal{F}$ can always be extended to the weighted facets $\tilde{\mathcal{F}}$, but this extension is not unique.

Given a wsc $\Omega$ and a group $G$ acting on $\Omega$ we generally identify the group action with the group and denote them together as $(\Omega, G)$. Thus we will call $(\Omega, G)$ a wsc with group action on $[n]$. Meaning that $\Omega$ is defined on $[n]$ and the group action of $G$ on $[n]$ can be canonically extended to group action on $\Omega$. 

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Note that since we consider wsc on finitely many vertices, the groups considered are always finite. In particular, they are subgroups of the full permutation group, which we denote $S_n$.

Given $(\Omega, G)$, we can extend the concept of refinement of wsc, introduced in Definition 2.4, to the wsc with group action. We say that $(\Omega', G)$ is a refinement of $(\Omega, G)$ if $\Omega'$ is a refinement of $\Omega$ and there exists a map $\pi : \tilde{F}' \to \tilde{F}$ such that $c' = c \circ \pi$. That is, there is a group action of $G$ on $\Omega'$ that reduces to the group action on $\Omega$ when not considering the weights.

### 2.1.3 Physical examples

In this section we explain how to interpret $(\Omega, G)$ as a characterisation of physical spaces. We look at different known Hamiltonians and construct a suitable $(\Omega, G)$ description from them. Note that all $\Omega$ in this setting will be non weighted simplicial complexes. That is because the weights are a mathematical tool that allow us to turn any $(\Omega, G)$ into a refinement $\Omega'$ of $\Omega$ such that there exists an extended action from $G$ on $\Omega'$ that is free on $\Omega'$, see Proposition 2.4. Freeness is then an important condition due to Theorem 2.2. This is why we consider weighted simplicial complexes instead of regular simplicial complexes. This is not reflected in the physical structure of our physical systems and therefore plays no role in this section. Regardless, we give refinements that allow free actions.

**Remark 2.1.** We are interested in studying decompositions of density matrices, not Hamiltonians. These examples are given with Hamiltonians because it is easier to see how physical properties translate into the $(\Omega, G)$ structure, and show that this structure (minus the weights) is very natural in physical systems.

Consider a physical system composed of $n + 1$ $d$-dimensional subsystems (we typically refer to each of these subsystems as a particle) evolving under a Hamiltonian $H$. The main idea of this section is that each particle is represented by a vertex in $\Omega$, each multiparticle interaction is modelled by a facet and the symmetries of $H$ are encoded in the group $G$. This construction is summarised in Table 2.1.

<table>
<thead>
<tr>
<th>Feature of $H$</th>
<th>Feature of $(\Omega, G)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Particles</td>
<td>Vertices of $\Omega$</td>
</tr>
<tr>
<td>Multiparticle terms</td>
<td>Facets of $\Omega$</td>
</tr>
<tr>
<td>Symmetries</td>
<td>$G$</td>
</tr>
</tbody>
</table>

Table 2.1: Properties of a Hamiltonian $H$ as they are encoded in an $(\Omega, G)$ pair. Note that the weighted structure of $\Omega$ is not reflected in the Hamiltonian.

**Example 2.5** (1D Ising model). Consider $n + 1$ particles interacting under a 1D quantum Ising model with periodic boundary conditions, that is $n + 1$ two dimensional particles in a 1D lattice with periodic boundary conditions, evolving under the Hamiltonian

$$H = -\sum_{i, i+1 \in [n]} J_{i,i+1} \sigma_z^i \sigma_z^{i+1} - \mu \sum_{i \in [n]} h_i \sigma_z^i$$
where $\sigma^z_i$ is the Pauli matrix in the $z$ direction that acts on the $i$th particle, $i + 1$ is taken mod $n + 1$, $J_{i,i+1}$ is the interaction strength of particles $i$ and $i + 1$, $\mu$ is the magnetic moment and $h_i$ is the external magnetic field at lattice site $i$.

Firstly, since we have $n + 1$ particles our simplicial complex $\Omega$ is on $[n]$. The second term of the equation does not contribute to $\Omega$ because it involves a single particle for each summand. The first term, on the other hand, is composed of two particle interactions ($J_{i,i+1} \neq 0$ for interacting systems). Thus we obtain that $\Omega(\{i,i+1\}) = 1$ $\forall i \in [n]$. We obtain that $\Omega$ is the circle of Fig. 2.7, which we denote $\Theta_n$.

![Figure 2.7: Simplicial complex $\Theta_n$, the circle.](image)

The general Ising model has no symmetries, so $G = \{e\}$. That said, there are several interesting cases we can consider.

i) Let $h_i = h$ $\land$ $J_{i,i+1} = J$ $\forall i \in [n]$. This allows us to use any group action that preserves the structure of $\Omega$. In particular consider the cyclic group of $n + 1$ with generator $c$ and neutral element $e$, that is $C_n = \{e, c, c^2, \ldots, c^n\}$ with group action on $i \in [n]$ defined as

$$c \cdot i = i + 1.$$

This is one of the cases covered in Example 2.2. This action is also clearly free on $\tilde{F}$. We call $(\Theta_n, C_n)$ the translational invariant circle.

ii) Let $n+1$ be even and $m = (n+1)/2$, and denote $[n] = \{-m, -(m-1), \ldots, -1, 1, 2, \ldots, m\}$. Let $h_{-i} = h_i$ $\forall i = 1, \ldots, m$ and $J_{i,i+1} = J_{-(i+1),-i}$ $\forall i = 1, \ldots, m - 1$. For this case we consider open boundary conditions, that is $J_{-m,m} = 0$. The Ising model Hamiltonian under these restrictions is

$$H = -J_{-1,1}\sigma^z_{1}\sigma^z_{-1} \sum_{i=1}^{m-1} J_{i,i+1}(\sigma^z_i\sigma^z_{i+1} + \sigma^z_{-(i+1)}\sigma^z_{-i}) - \mu \sum_{i=1}^{m} h_i(\sigma^z_{-i} + \sigma^z_{-i}).$$

Since one interaction term is now 0 we obtain a different simplicial complex: the line in Fig. 2.8 which we denote be $\Lambda_n$.

![Figure 2.8: Simplicial complex $\Lambda_n$, the line.](image)

This Hamiltonian has a reflection symmetry. This is represented by the two element group $G = \{e, g\}$ with the action

$$g \cdot i = -i.$$
Note that the induced action on \( \tilde{F} = F^4 \) is not free, since \( g \cdot \{-1, 1\} = \{-1, 1\} \).

We can fix that by adding weight to the simplex. In particular let

\[
\Omega'(\{i, j\}) = \begin{cases} 2 & \text{if } \{i, j\} = \{-1, 1\} \\ \Omega(\{i, j\}) & \text{if } \{i, j\} \neq \{-1, 1\}, \end{cases}
\]

i.e. the weight of the facet \( \{-1, 1\} \) is raised by 1. Now we define the following extension of the action on the elements \((F, k) \in \tilde{F}\). Note that \( k = 0 \) if \( F \neq \{-1, 1\} \) and \( k = 0, 1 \) if \( F = \{-1, 1\} \). This extension is

\[
g \cdot (F, k) = \begin{cases} (F, k + 1 \pmod{2}) & \text{if } F = \{-1, 1\} \\ (g \cdot F, k) & \text{if } F \neq \{-1, 1\}. \end{cases}
\]

This action is free on \( \tilde{F} \), and therefore free on \( \Omega' \). The weighted simplicial complex \( \Omega' \) is, for the case \( n + 1 = 6 \), shown in Fig. 2.9.

Figure 2.9: Simplicial complex \( \Omega' \) for \( n + 1 = 6 \).

iii) Consider a similar case to the previous one, but now with \( n + 1 \) odd. Denote \( m = n/2 \) and the elements of \([n] \) as \([n] = \{-m, -(m - 1), \ldots, -1, 0, 1, \ldots, m\} \).

Consider the Hamiltonian

\[
H = - \sum_{i=0}^{m-1} J_{i,i+1}(\sigma_z^i \sigma_z^{i+1} + \sigma_z^{-(i+1)} \sigma_z^{-i}) - \mu h_0 \sigma_z^0 - \mu \sum_{i=1}^{m} h_i (\sigma_z^i + \sigma_z^{-i}),
\]

as before, the simplicial complex associated to \( H \) is \( \Lambda_n \) (Fig. 2.8). This Hamiltonian also has the same symmetry: let \( G = \{e, g\} \) with action on \( [n] \)

\[
g \cdot i = -i.
\]

Note that \( g \cdot 0 = 0 \), so this action is not free on \([n]\), but the induced group action on \( \tilde{F} = F \) is free. Thus we obtain an \((\Omega, G)\) such that the action of \( G \) on \( \Omega \) is free, but not the action on \([n]\).

The Ising model only deals with Hamiltonians with two particle interactions, which are common in physics because they are more tractable that higher order interactions, but higher order interactions can be found in physics. Here we apply our description to the model described in \([19]\). Note that the \( \Omega \) derived from the Hamiltonian also appers in their analysis, as seen in \([19, \text{Figure 1}]\).

\(^4\)We are using an abuse of notation here since the elements \( \tilde{F} \) and \( F \) are different objects. In the case where \( \Omega \) is a simplicial complex these 2 sets become bijective through the collapse map, which justifies the equality.
Example 2.6. Consider \( n+1 \) spin-\( S \) particles in 1D interacting under 3-spin next-nearest neighbour interactions. This is shown in the Hamiltonian

\[
H = J_1 \sum_{i \in [n]} (1 - \delta(-1)^i) \vec{S}_i \cdot \vec{S}_{i+1} + J_2 \sum_{i \in [n]} \vec{S}_{i-1} \cdot \vec{S}_{i+1} + J_3 \sum_{i \in [n]} [(\vec{S}_{i-1} \cdot \vec{S}_i)(\vec{S}_i \cdot \vec{S}_{i+1}) + \text{h.c.}],
\]

where \( \text{h.c.} \) denotes Hermitian conjugate. The element \( \vec{S}_i \) is the spin operator \( \vec{S} = (S_x, S_y, S_z) \) associated to particle \( i \). The general structure of the simplicial complex \( \Omega \) derived from this Hamiltonian is shown in Fig. 2.10. We consider only periodic boundary conditions, in particular the following distinct cases:

\[\cdots\]
\[i\]
\[i+1\]
\[i+2\]
\[i+3\]
\[\cdots\]

Figure 2.10: Structure of \( \Omega \) far from 0, \( n \). The even vertices are on top and the odd vertices on the bottom, or vice versa.

i) Let \( n+1 \) be even. Then the band in Fig. 2.10 has the global structure shown in Fig. 2.11. This structure has two distinct edges, which correspond to even and odd vertices. We can consider the cyclic group \( C_n \) with generator \( c \) acting as \( c \cdot i = i + 1 \) if we have translational invariance in the Hamiltonian.

\[\cdots\]
\[i\]
\[i+1\]
\[i+2\]
\[i+3\]
\[\cdots\]

Figure 2.11: Topological structure of the simplicial complex resulting from an even number of vertices. The simplicial structure is inherited from Fig. 2.10.

ii) Let \( n+1 \) be odd. In this case the parity of a given vertex is not clear if we consider \( \text{mod } n+1 \) addition. To see that consider \( n \), which is even. On the other hand, \( 2n = n - 2 \text{ mod } n+1 \) is odd. Thus we can not split vertices into even and odd, and the band turns into a Möbius strip, since we need a band with a single edge.

2.2 Tensor decomposition over weighted simplicial complex with group action

From this point onwards \( (\Omega, G) \) is a simplicial complex over \( [n] \) with group action. We want to consider tensors on \( (\Omega, G) \) in the following sense:
Example 2.7. Consider the tensor rank decomposition [11] of a tensor

\[ v \in \bigotimes_{i \in [n]} V_i = \mathcal{V}, \]

where each \( \mathcal{V} \) is a vector space. That is

\[ v = \sum_{\alpha = 1}^{r} a_\alpha^{[0]} \otimes a_\alpha^{[1]} \otimes \cdots \otimes a_\alpha^{[n]}, \]

where \( a_\alpha^{[i]} \in V_i \). The minimum \( r \) such that this decomposition exists is the tensor rank, denoted \( TR(v) \). The structure of this decomposition is conveyed by the full \( n \) simplex \( \Sigma_n \). To each vertex we assign a vector space \( V_i \) and to the only facet of this simplex, \( F = \{0, 1, \ldots, n\} \), we assign the index \( \alpha \). Since \( F \) contains all vertices, \( \alpha \) indexes all local tensors \( a^{[i]} \). If the tensor fulfils any symmetry, we can model it through a group \( G \). In this case, if \( g \in G \) we require that

\[ g \cdot v = \sum_{\alpha = 1}^{r} a_{g \cdot \alpha}^{[0]} \otimes a_{g \cdot \alpha}^{[1]} \otimes \cdots \otimes a_{g \cdot \alpha}^{[n]}, \]

\( g \cdot \alpha \) is the action of \( g \) on \( F \), since we assigned \( \alpha \) to this facet. For this to be well defined we require that

\[ \exists g \in G \mid g \cdot i = j \Rightarrow V_i = V_j. \]

This construction is generalised to arbitrary \( (\Omega, G) \) in Definition 2.10.

Following the example, assign to each vertex of an \( (\Omega, G) \) a local \( \mathbb{C} \)-vector space \( V_i \). Moreover if \( \exists g \in G \) such that \( g \cdot i = j \), then \( V_i = V_j \). The global vector space is the tensor product of the local vector spaces:

\[ \mathcal{V} = \bigotimes_{i \in [n]} V_i. \]

Let \( \mathcal{I} \) be an index set (if \( \mathcal{I} \) is finite, think of \( [r] \)). The set of maps from \( \tilde{F} \) to \( \mathcal{I} \) is denoted \( \mathcal{I}^{\tilde{F}} \). We denote the elements of this set by \( \alpha \). The restriction of \( \alpha \in \mathcal{I}^{\tilde{F}} \) to \( \tilde{F}_i \), \( \alpha_{ij} \in \mathcal{I}^{\tilde{F}_i} \) is denoted \( \alpha_{ij} \in \mathcal{I}^{\tilde{F}_i} \).

In the upcoming section we define three different decompositions for elements of \( \mathcal{V} \) on wcsc with group action \( (\Omega, G) \).

### 2.2.1 The invariant decomposition

In this section we define the invariant decomposition, we provide some examples and prove some existence results.
Definition 2.10 ((\(\Omega, G\)) decomposition). Consider \((\Omega, G)\), a vector space \(V\) composed of local vector spaces on \((\Omega, G)\) and let \(v \in V\). An \((\Omega, G)\) decomposition of \(v\) consists of families of local vectors 
\[
\left( V^{[i]}_{\alpha_i} \right)_{\alpha \in \tilde{F}}
\]
with \(V^{[i]}_{\alpha_i} \in V_i \forall i \in [n]\) such that
\[
v = \sum_{\alpha \in \tilde{F}} V^{[0]}_{\alpha_0} \otimes V^{[1]}_{\alpha_1} \otimes \cdots \otimes V^{[n]}_{\alpha_n}
\]
and \(\forall i \in [n], g \in G\) and \(\alpha \in \tilde{F}\) we have
\[
V^{[i]}_{\alpha_i} = V^{[g \cdot i]}_{\sigma(a_i)}.
\]

The \((\Omega, G)\) rank of \(v\), denoted by \(\text{rank}_{(\Omega, G)}(v)\), is the minimum cardinality of \(\tilde{I}\) such that this decomposition exists. If no such decomposition exists the \((\Omega, G)\) rank is taken to be \(\infty\). If \(G = \{e\}\), the trivial group then we denote \(\text{rank}_{(\Omega, G)}(v)\) by \(\text{rank}_\Omega(v)\).

This definition formalises what we wanted from Example 2.7; notably, the identification of facets \(F \in \tilde{F}\) with index elements. The natural way to do this is to consider, for each sumand, a mapping

\[ F \mapsto \alpha_F \mapsto k \]

where \(F \in \tilde{F}\), \(\alpha_F\) the index associated to facet \(F\) and \(k \in I\) and the first map is bijective. In the formalism of Definition 2.10 the first map is ignored, because it is a bijection, and we consider maps \(\tilde{F} \rightarrow I\) (i.e. elements of \(\tilde{I}\)).

Note that the \((\Omega, G)\) decomposition naturally imposes the structure and symmetries of \((\Omega, G)\). The first condition asks for a decomposition that respects the structure of \(\Omega\): the local vector space elements associated to a vertex \(i\) share indices with the local vector space elements which share weighted facets with \(i\). How many indices are shared and which other local vector space elements also share it is determined by the weighted facet structure of \(\Omega\). The second condition specifies how the group action affects the local vectors. The \((\Omega, G)\) rank is the number of local vectors needed to write an element in this form. This explanation should become clear with the following examples:

Example 2.8. i) Consider Example 2.7. With the new formalism from Definition 2.10, let \(\Omega = \Sigma_n\), the \(n\) full simplex. The set of facets has a single element \(F\). Then a \((\Sigma_n, G)\) decomposition is

\[ v = \sum_{\alpha \in \tilde{F}} V^{[0]}_{\alpha_0} \otimes V^{[1]}_{\alpha_1} \otimes \cdots \otimes V^{[n]}_{\alpha_n} = \sum_{\alpha \in \tilde{I}} V^{[0]}_{\alpha} \otimes V^{[1]}_{\alpha} \otimes \cdots \otimes V^{[n]}_{\alpha}. \]

Note that for the second equality we have taken the implicit bijection between facets and indices. Moreover, if we assume the rank is finite, then

\[ v = \sum_{\alpha=1}^r V^{[0]}_{\alpha} \otimes V^{[1]}_{\alpha} \otimes \cdots \otimes V^{[n]}_{\alpha}. \]
Thus we recover the standard definition of tensor rank decomposition. Consider now translational invariance, that is the cyclic group $C_n$ generated by $c$ acting as
\[ c \cdot i = i + 1 \pmod{n+1} \quad \forall i \in [n]. \]
This group leaves a single orbit, thus set $V_i = V_0 \forall i \in [n]$. From the second condition in Definition 2.10 we get $\forall i, j \in [n]$
\[ v^{[i]}_\alpha = v^{[j]}_\alpha \]
by considering the group element $c^{j-i}$ acting on $v^{[i]}_\alpha$. Since this can be done for all $i, j \in [n]$, we obtain that the local tensors are all equal. Thus the translationally invariant decomposition looks like
\[ v = \sum_{\alpha=1}^r V_\alpha \otimes V_\alpha \otimes \cdots \otimes V_\alpha. \]
Note that this decomposition is also fully symmetric, thus
\[ \text{rank}_{\Sigma_n, C_n}(v) = \text{rank}_{\Sigma_n, S_n}(v) \]
for all fully symmetric $v$, where $S_n$ is the full permutation group over $n + 1$ elements.

ii) Consider now the complete graph over 4 vertices, which we denote by $K_3$. The set of weighted facets is the set of edges of the graph, which we denote as shown in Fig. 2.12.

![Figure 2.12: Complete 3 graph, $K_3$, with labeled vertices and edges.](image)

A decomposition over this $\Omega$, with the trivial group action, looks like
\[ v = \sum_{\alpha,\beta,\gamma,\delta,\epsilon,\zeta \in I} V^{[0]}_{\alpha\beta\zeta} \otimes V^{[1]}_{\alpha\gamma\epsilon} \otimes V^{[2]}_{\gamma\delta\zeta} \otimes V^{[3]}_{\delta\beta\epsilon}. \]
If we add the full permutation group acting canonically as
\[ \sigma \cdot i = \sigma(i) \]
∀σ ∈ Sn, i ∈ [n]. We obtain the additional conditions that each family of local vectors is equal, denoted by V, and with the property that each Vαβζ is fully symmetric. This (K₃, S₃) decomposition has the same symmetry as the full tensor rank decomposition (for the case n = 3), showing that the simplicial complex is a relevant part to encode the structure of the decomposition of an element v ∈ V.

iii) Consider the circle, Θₙ, as in Fig. 2.7. Denote each facet, that is each edge, by the number associated to the vertex on its left. Then a (Θ, {e}) decomposition is

\[ v = \sum_{\alpha_0, \ldots, \alpha_n \in I} V^{[0]}_{\alpha_0} \otimes \cdots \otimes V^{[n]}_{\alpha_n - 1} \alpha_n. \]

The translationally invariant version is

\[ v = \sum_{\alpha_0, \ldots, \alpha_n \in I} V_{\alpha_n \alpha_0} \otimes \cdots \otimes V_{\alpha_n - 1 \alpha_n}, \]

with the only change being that the local tensors now need to be equal.

Note that these decompositions had been studied before under a less general formulation [6]. In particular we have the following:

\[ osr(v) = \text{rank}_{\Theta_n}(v) \]
\[ t.i.-osr(v) = \text{rank}_{(\Theta_n, C_n)}(v), \]

where osr(v) and t.i.-osr(v) are notation used in [6]. This example shows that the framework presented in this work generalises these concepts.

iv) Let n = 1 and consider the edge and double edge in Fig. 2.13. Consider the nontrivial action of the 2 element group, which can be taken free over the double edge. The decompositions of these (Ω, G) are

\[ v = \sum_{\alpha \in I} V_\alpha \otimes V_\alpha \]
\[ v = \sum_{\alpha, \beta \in I} V_{\alpha \beta} \otimes V_{\beta \alpha}. \]

Figure 2.13: Single edge (left) and double edge (right).

These particular decompositions were studied in [6] without having developed the theoretical framework of the (Ω, G) decomposition.
v) Let \( n = 3 \) and consider Example 2.4, the wsc \( \omega \) is shown again in Fig. 2.14. A tensor decomposition over this wsc is

\[
v = \sum_{a,b \in I} V^{[0]}_a \otimes V^{[1]}_a \otimes V^{[2]}_{a,b} \otimes V^{[3]}_b.
\]

![Figure 2.14: An example of a simplicial complex.](image)

If we consider the 2 element group \( C_1 = \{e, g\} \) as in Example 2.4, the associated \((\Omega, C_1)\) decomposition is

\[
v = \sum_{a,b \in I} V^{[0,1]}_a \otimes V^{[0,1]}_a \otimes V^{[2]}_{a,b} \otimes V^{[3]}_b,
\]

where we only require the local tensors on 0 and 1 to be equal. To make this action free we showed in Example 2.4 that we need to double the weight on each facet. Call this new wsc \( \Omega' \), the \((\Omega', C_1)\) decomposition is

\[
v = \sum_{a_1,a_2,b_1,b_2 \in I} V^{[0,1]}_{a_1,a_2} \otimes V^{[0,1]}_{a_2,a_1} \otimes V^{[2]}_{a_1,a_2,b_1,b_2} \otimes V^{[3]}_{b_1,b_2},
\]

with the additional constraints that

\[
V^{[2]}_{a_1,a_2,b_1,b_2} = V^{[2]}_{a_2,a_1,b_1,b_2} \quad V^{[3]}_{b_1,b_2} = V^{[3]}_{b_2,b_1}.
\]

We now proceed to show how the rank behaves under addition and (when possible) multiplication.

**Proposition 2.2.** Let \((\Omega, G)\) be a connected wsc with group action on \([n]\). Then for all \( v, w \in V \)

i) \( \rank_{(\Omega,G)}(v + w) \leq \rank_{(\Omega,G)}(v) + \rank_{(\Omega,G)}(w) \).

ii) If each \( V_i \) has a multiplication defined,\(^5\) then

\[
\rank_{(\Omega,G)}(vw) \leq \rank_{(\Omega,G)}(v) \rank_{(\Omega,G)}(w).
\]

\(^5\)That is each \( V_i \) is an algebra.
Proof. Both statements are obvious if $v$ or $w$ do not have a finite $(\Omega, G)$ decomposition, so let us assume they do. Let
\[ V^{[i]}_{\beta_i}, \quad \beta_i \in \tilde{I}, \quad W^{[i]}_{\gamma_i}, \quad \gamma_i \in \tilde{J} \]
form an $(\Omega, G)$ decomposition of $v$ and $w$, respectively, and let \( \text{rank}_{(\Omega,G)}(v) = |I| \), \( \text{rank}_{(\Omega,G)}(w) = |J| \).

For $i)$, take the direct sum of the local tensors. That is, let \( L = I \sqcup J \) be the disjoint union of the index sets. Note that \( \tilde{I} \sqcup \tilde{J} \subset \tilde{L} \) (and also holds for \( J \)). Thus, set \( \alpha \in L \), it makes sense to define the following local tensors:
\[
A^{[i]}_{\alpha_i} = \begin{cases} 
V^{[i]}_{\alpha_i} & \text{if } \alpha_i(F) \in I \quad \forall F \in \tilde{F}_i \\
W^{[i]}_{\alpha_i} & \text{if } \alpha_i(F) \in J \quad \forall F \in \tilde{F}_i \\
0 & \text{else.}
\end{cases}
\]

First, we check the second condition of Definition 2.10. Let \( g \in G \), then
\[
A^{[g]}_{s(\alpha_i)} = \begin{cases} 
V^{[g]}_{s(\alpha_i)} & \text{if } g\alpha_i(F) \in I \quad \forall F \in \tilde{F}_{gi} \\
W^{[g]}_{s(\alpha_i)} & \text{if } g\alpha_i(F) \in J \quad \forall F \in \tilde{F}_{gi} \\
0 & \text{else.}
\end{cases}
\]

\( g\alpha_i(F) \in I \forall F \in \tilde{F}_{gi} \) is equivalent to itself without the $g$. To show that, let \( i \in \llbracket n \rrbracket \) note that every element in \( \tilde{F}_{gi} \) can be written as $gF$ with $F \in \tilde{F}_i$. Thus
\[
g\alpha_i(gF) \in I \quad \forall gF \in \tilde{F}_{gi} \quad \Rightarrow \quad \alpha_i(g^{-1}gF) = \alpha_i(F) \in I \quad \forall F \in \tilde{F}_i.
\]

The same argument works for the second condition of the definition of \( A^{[g]}_{s(\alpha_i)} \). Then, note that since \( V^{[i]}_{\alpha_i} \) and \( W^{[i]}_{\alpha_i} \) are local tensors of an $(\Omega, G)$ decomposition
\[
V^{[g]}_{s(\alpha_i)} = V^{[i]}_{\alpha_i}, \quad W^{[g]}_{s(\alpha_i)} = W^{[i]}_{\alpha_i},
\]
thus
\[
A^{[g]}_{s(\alpha_i)} = A^{[i]}_{\alpha_i},
\]
which is the second condition of Definition 2.10.

Let \( \alpha \in \tilde{L} \). Note that if there exists \( F, F' \in \tilde{F} \) such that \( \alpha(F) \in I \) but \( \alpha(F') \notin I \), by connectedness of \( \Omega \), there will be a local tensor \( A^{[i]}_{\alpha_i} = 0 \) for some \( i \in \llbracket n \rrbracket \). The same argument works for \( J \). Therefore the nonzero elements of the sum are
\[
\sum_{\alpha \in \tilde{L}} A^{[0]}_{\alpha_0} \otimes \cdots \otimes A^{[n]}_{\alpha_n} = \sum_{\alpha \in \tilde{I}} A^{[0]}_{\alpha_0} \otimes \cdots \otimes A^{[n]}_{\alpha_n} + \sum_{\alpha \in \tilde{J}} A^{[0]}_{\alpha_0} \otimes \cdots \otimes A^{[n]}_{\alpha_n},
\]
but these are just the conditions in the definition of the local tensors, therefore this is $v+w$ and
\[
\text{rank}_{(\Omega,G)}(v+w) \leq |L| = |I| + |J| = \text{rank}_{(\Omega,G)}(v) + \text{rank}_{(\Omega,G)}(w).
\]

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To prove ii), consider the cartesian product of the index sets, that is \( L = I \times J \), and the canonical projections \( \pi_I : L \to I \) and \( \pi_J : L \to J \).

Let \( \alpha \in L \setminus F \) and define the new local tensors as

\[
A_i^{0} | i = V_i^{0} \pi_I \circ \alpha | i \otimes \cdots \otimes V_i^{n} \pi_I \circ \alpha | i.
\]

Clearly \( A_i^{0} | i \) fulfills the second condition of Definition 2.10 because \( V_i^{0} \pi_I \circ \alpha | i \) and \( W_i^{0} \pi_J \circ \alpha | i \) fulfill it. For the first condition of Definition 2.10, note that \( \alpha \in L \setminus F \) if and only if

\[
\sum_{\alpha \in L \setminus F} V_i^{0} \pi_I \circ \alpha | i \otimes \cdots \otimes W_i^{n} \pi_J \circ \alpha | i = 0.
\]

Then

\[
\text{rank}_{G \otimes F}(\Omega_{ F}, G \otimes F)(vw) \leq |L| = |I||J| < \text{rank}_{G \otimes F}(\Omega_{ F}, G \otimes F)(v) \text{ rank}_{G \otimes F}(\Omega_{ F}, G \otimes F)(w),
\]

which proves the claim.

### 2.2.2 Finite decompositions

#### Necessary conditions

Consider the following definition:

Definition 2.11. Let \((\Omega, G)\) be a wsc with group action over \([n]\) and \(V\) its associated global vector space. The invariant subspace \(V_{\text{inv}}\) is the set of \(v \in V\) such that

\[
g \cdot v = v,
\]

where \(g \cdot v = \sum_{\alpha \in I \setminus \tilde{F}} V_0^{\alpha} \alpha | 0 \otimes \cdots \otimes V_n^{\alpha} \alpha | n\).

Note that we assumed that an invariant \(v \in V_{\text{inv}}\) admits a decomposition to define the action of \(g\) on \(v\). We see in Theorem 2.2 that \(v \in V_{\text{inv}}\) is a necessary condition for a finite \((\Omega, G)\) decomposition. First, we can show that it is also a sufficient condition:

Proposition 2.3. Let \(v \notin V_{\text{inv}}\). Then there exists no finite \((\Omega, G)\) decomposition of \(v\).
Proof. Consider the contrapositive statement: let \( v \) have an \((\Omega, G)\) decomposition, then \( v \in \mathcal{V}_{\text{inv}} \). We show that this is true.

Let \( v \in \mathcal{V} \) and

\[
v = \sum_{\alpha \in \mathcal{I}} V_{\alpha|0} \otimes \cdots \otimes V_{\alpha|n}.
\]

Then

\[
g \cdot v = \sum_{\alpha \in \mathcal{I}} V_{\alpha|0}^g \otimes \cdots \otimes V_{\alpha|n}^g \\
= \sum_{\alpha \in \mathcal{I}} V_{\alpha|0}^{g\left((g^{-1}\alpha)_{|0}\right)} \otimes \cdots \otimes V_{\alpha|n}^{g\left((g^{-1}\alpha)_{|n}\right)} \\
= \sum_{\alpha \in \mathcal{I}} V_{\alpha|0}^{g(0)} \otimes \cdots \otimes V_{\alpha|n}^{g(n)} \\
= \sum_{\alpha \in \mathcal{I}} V_{\alpha|0}^{g(0)} \otimes \cdots \otimes V_{\alpha|n}^{g(n)} = v.
\]

In the third equality we used the second condition of an \((\Omega, G)\) decomposition, and on the third equality we used that \( g^{-1} \) runs through \( \mathcal{I}^\mathcal{F} \) if \( \alpha \) does, i.e. \( \forall \alpha \in \mathcal{I}, \exists g \in G, \alpha' \in \mathcal{I}^\mathcal{F} \) such that

\[
g^{-1} \alpha' = \alpha.
\]

To see that the fist equality is true, first note that it is enough to show that for all \( i \in [n] \)

\[
\alpha|_{gi} = g\left(\left(g^{-1}\alpha\right)_{|i}\right).
\]

The left hand side can be explicitly written as

\[
\alpha|_{gi} : \tilde{F}|_{gi} \rightarrow \mathcal{I} \\
\mathcal{F} \mapsto \alpha(F).
\]

We only need to show that the right hand side reduces to the same expression. Let us start with \( \alpha : \tilde{F} \rightarrow \mathcal{I} \) and add each element step by step:

i) Adding \( g^{-1} \) as described in Definition 2.7 (note that \( g^{-1}\tilde{F} = g\tilde{F} = \tilde{F} \)) gives

\[
g^{-1} \alpha : \tilde{F} \rightarrow \mathcal{I} \\
\mathcal{F} \mapsto \alpha(gF).
\]

ii) The restriction to \( i \) only changes the domain set to

\[
\left(g^{-1}\alpha\right)|_{i} : \tilde{F}_{i} \rightarrow \mathcal{I} \\
\mathcal{F} \mapsto \alpha(gF')
\]

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iii) Finally, the addition of $g$ changes the domain set and the image according to Definition 2.7 as
\[
g \left( (g^{-1} \alpha) \right) : g\tilde{F}_i \rightarrow \mathcal{I}
\]
\[
F \mapsto (g^{-1} \alpha) \big|_{\tilde{F}_i} (g^{-1} F) = \alpha (gg^{-1} F),
\]
but this is just
\[
g \left( (g^{-1} \alpha) \right) : \tilde{F}_{g_i} \rightarrow \mathcal{I}
\]
\[
F \mapsto \alpha(F),
\]
which is what we wanted. The Proposition follows.

This Proposition shows that it only makes sense to talk about the $(\Omega, G)$ decompositions in finite terms for tensors that are invariant under $G$. The next question we answer is the converse. Given tensors in $\mathcal{V}_{inv}$, what can we say about the existence of a finite $(\Omega, G)$ decomposition?

**Sufficient conditions**

First, consider the case with trivial action:

**Theorem 2.1.** Let $\Omega$ be a connected wsc on $[n]$. Let $v \in \mathcal{V}$, then
\[
\text{rank}_{\Omega}(v) < \infty.
\]

**Proof.** Start with the tensor rank decomposition of $v$, that is $v$ written as
\[
v = \sum_{\alpha \in \mathcal{I}} w_{\alpha}^{[0]} \otimes \cdots \otimes w_{\alpha}^{[n]}
\]
with $|\mathcal{I}| < \infty$. The tensor rank is finite for all elements in a tensor product space, so this can be done $\forall v \in \mathcal{V}$ [11]. For $i \in [n]$ consider $\beta \in \mathcal{I}$ define a new set of local vectors
\[
V_{\beta_i}^{[i]} = \begin{cases} 
w_{\alpha}^{[i]} & \text{if } \beta_i(F) = \alpha \forall F \in \tilde{F}_i, \\
0 & \text{else.}
\end{cases}
\]
Consider the decomposition that corresponds to the new local vectors $V_{\beta_i}^{[i]}$:
\[
\sum_{\beta \in \mathcal{I}} V_{\beta_0}^{[0]} \otimes \cdots \otimes V_{\beta_n}^{[n]}.
\]
Because $\Omega$ is connected, each $\beta_i$ will be constant with value $\alpha$ if and only if $\beta$ is the constant function with value $\alpha$ over the whole $\tilde{F}$. Thus we can write the nonzero
sumands as a summation over \( \alpha \), changing the local vectors \( V^{[i]}_{\beta} \) by their defined value \( w^{[i]}_{\alpha} \). Therefore

\[
\sum_{\beta \in I} V^{[0]}_{\beta} \otimes \cdots \otimes V^{[n]}_{\beta} = \sum_{\alpha \in I} w^{[0]}_{\alpha} \otimes \cdots \otimes w^{[n]}_{\alpha} = v,
\]

which gives a finite \((\Omega, \{e\})\) decomposition of \( v \), proving the result.

This Theorem does not involve any nontrivial group action, but to achieve a similar result for the general case we first need the following result regarding \((\Omega, G)\). Recall that a refinement is a wsc on the same vertices and with the same facet structure, but higher weights.

**Proposition 2.4.** Let \((\Omega, G)\) be a wsc with group action on \([n]\). Then there exists a refinement of \( \Omega, \Omega' \), such that there exists a free action from \( G \) on \( \Omega' \) induced from the same action on \([n]\).

**Proof.** Let \( r = |G| \) and \( G = \{g_1, \ldots, g_r\} \). Define \( \Omega' \) as

\[
\Omega'(S) = \begin{cases} 
\Omega(S) & \text{if } S \notin F \\
r\Omega(S) & \text{if } S \in F.
\end{cases}
\]

\( \Omega' \) is clearly a refinement of \( \Omega \). For \( F \in F \) with \( \Omega(F) = m \), denote by

\( F_1, \ldots, F_m \)

the weights of \( F \). Moreover, denote the \( rm \) weights of \( F \) in \( \Omega' \) as

\( F^{g_1}_{1}, \ldots, F^{g_r}_{1}, \ldots, F^{g_1}_{m}, \ldots, F^{g_r}_{m} \).

The collapse map of \( \Omega' \), \( c' : \tilde{F}' \to F' \) is defined through the collapse map of \( \Omega \) and the map

\[
\pi : \tilde{F}' \longrightarrow \tilde{F} \\
F^{g}_{i} \longmapsto F_{i}.
\]

Then \( c' \) is defined through the commuting diagram in Fig. 2.15

![Figure 2.15: Commutative diagram to define \( c' \), the collapse map of \( \Omega \).](image)

Finally, let \( i \in [n], g, h \in G \). The action of \( G \) on \( \tilde{F}' \) is

\[
g \cdot F^{h}_{i} = (gF_{i})^{gh}.
\]

This is clearly an action and since \( gh \neq h \ \forall g \neq e \), this action is free. Moreover, from the definition we see immediately that \( \pi \) is \( G \)-linear, thus this action on \( \Omega' \) is a refinement of the action on \( \Omega \). \( \square \)
Proposition 2.4 says that given an \((\Omega, G)\), we can raise the weights of \(\Omega\) to make the action free. In particular, if we are only concerned with the nonweighted facet structure, \(\mathcal{F}\), this states that any action is free. We already saw that for the particular case of Example 2.5. This result becomes important due to the following existence theorem, which says that a finite \((\Omega, G)\) decomposition exists for invariant tensors if the group action is free.

**Theorem 2.2.** Let \((\Omega, G)\) be a connected wsc with group action on \([n]\), \(\mathcal{V} = \mathcal{V}_0 \otimes \mathcal{V}_n\) its associated vector space and let \(v \in \mathcal{V}_{inv}\). Then

\[
\text{rank}_{(\Omega,G)}(v) < \infty.
\]

To prove this Theorem we first need the following Lemma for general group actions:

**Lemma 2.1.** Let a group \(G\) act on a set \(X\). The action of \(G\) on \(X\) is free if and only if there exists a \(G\)-linear map \(z: X \rightarrow G\), where \(G\) acts on itself by left multiplication \((g \cdot h = gh)\).

**Proof.** First, let \(G\) act freely on \(X\). 2 elements \(x, y \in X\) are in the same orbit if there exists \(g \in G\) such that \(g \cdot x = y\). Clearly, being in the same orbit is an equivalence relation due to the definition of group action. Thus we can split \(X\) into orbits in a well defined, unique way. Let \(X\) have \(k\) orbits and choose a representative \(x_l\), \(l = 1, \ldots, k\) for each orbit. For all \(x \in X\) there exists a unique pair \(g \in G, l = 1, \ldots, k\) such that \(x = g \cdot x_l\). Define \(z\) as \(z(x) = z(g \cdot x_l) = g\). To show that \(z\) is \(G\)-linear, let \(x \in X, h \in G\). First, we know that \(x = g \cdot x_l\) by definition. Then

\[
z(h \cdot x) = z(h \cdot (g \cdot x_l)) = z(hg \cdot x_l) = hg = hz(g \cdot x_l) = hz(x),
\]

which proves \(G\)-linearity of \(z\) and thus the first implication.

For the second, let \(z: X \rightarrow G\) be \(G\)-linear and the action of \(G\) on \(X\) be not free. Then there exist \(g \in G, x \in X\) such that \(g \neq e\) and \(g \cdot x = x\). Then

\[
gz(x) = z(g \cdot x) = z(x),
\]

but that is only possible if \(g = e\), which leads to contradiction. Thus the action of \(G\) on \(X\) is free, and the result follows. \(\Box\)

**Proof of Theorem 2.2.** Let \(G\) act freely on a wsc \(\Omega\). From Lemma 2.1 there exists

\[
z: \mathcal{F} \rightarrow G
\]

\(G\)-linear. For \(v \in \mathcal{V}_{inv}\), from Theorem 2.1, there exists an \((\Omega, \{e\})\) decomposition with finite rank. Denote the local tensors of this decomposition

\[
W_{\beta_1},
\]

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for $i \in [n]$, $\beta \in \vec{\mathcal{F}}$. Consider a new index set $\vec{\mathcal{I}} = \mathcal{I} \times G$ and the projection maps

$$\pi_\mathcal{I} : \vec{\mathcal{I}} \rightarrow \mathcal{I}, \quad \pi_G : \vec{\mathcal{I}} \rightarrow G.$$ 

Let $\alpha \in \vec{\mathcal{F}}$ and define local tensors for each $i \in [n]$ as

$$V[i]_{\alpha|i} = \begin{cases} W[g\alpha]_{\pi_\mathcal{I} \circ \alpha|i} & \text{if } \pi_G \circ \alpha|i = \left(g^{-1}z\right)_i \\ 0 & \text{else.} \end{cases}$$

To understand this definition, first look at the condition $\pi_G \circ \alpha|i = \left(g^{-1}z\right)_i$. Fix $\alpha \in \vec{\mathcal{F}}$, then

$$\pi_G \circ \alpha|i : \vec{\mathcal{F}}_i \rightarrow G, \quad F \mapsto g_{\mathcal{F}}.$$ 

On the other hand,

$$\left(g^{-1}z\right)_i : \vec{\mathcal{F}}_i \rightarrow G, \quad F \mapsto z(g \cdot F) = gz(F),$$

where $g_{\mathcal{F}}$ is notation for the image of $F$ by this map. Thus, the condition asks whether $\exists g \in G$ such that $gz(F) = g_{\mathcal{F}} \forall F \in \vec{\mathcal{F}}_i$. Note that for each $F$, this $g$ is uniquely defined as

$$g = z(F)^{-1}g_{\mathcal{F}}$$

therefore this will in general not be true (because it must hold for all $F$). Thus the need for the condition. We need to show that the subindex $g \left(\pi_\mathcal{I} \circ \alpha|i\right)$ is an element of $\vec{\mathcal{F}}_i$ to ensure the local vectors are well defined. Starting with $\alpha$, the image is $\mathcal{I} \times G$ and the domain $\vec{\mathcal{F}}$ with $\alpha$. The image remains unchanged with the restriction to $i$, but the domain changes to $\vec{\mathcal{F}}_i$. The projection $\pi_\mathcal{I}$ changes the image to $\mathcal{I}$ and leaves the domain unchanged. Finally, $g$ leaves the image unchanged, but changes the domain to $g_{\mathcal{F}} = \vec{\mathcal{F}}_i$ (as per Definition 2.7). Thus

$$g \left(\pi_\mathcal{I} \circ \alpha|i\right) : \vec{\mathcal{F}}_i \rightarrow \mathcal{I},$$

an element of $\vec{\mathcal{F}}_i$ as needed per Definition 2.10.

We now check that this local vectors form an $(\Omega, G)$ decomposition of $v$. Since $\mathcal{I}$ and $G$ are finite, then $\mathcal{I} \times G$ is finite. Thus we immediately obtain the result if these vectors form a decomposition. First we check that the local vectors fulfil the second condition, that is $\forall i \in [n], g \in G, \alpha \in \vec{\mathcal{I}}$

$$V[i]_{\alpha|i} = V[g\alpha]_{\pi_\mathcal{I} \circ \alpha|i}.$$
Let $h \in G$, then

$$V^{[hi]}_{h([\alpha_i])} = \begin{cases} W^{[gh]}_{\pi_G \circ h([\alpha_i])} & \text{if } \pi_G \circ ([\alpha_i]) = (g^{-1}z)_{hi} \\ 0 & \text{else.} \end{cases}$$

Note that

i) The following expression:

$$g \left( \pi_I \circ h ([\alpha_i]) \right) = gh \left( \pi_I \circ [\alpha_i] \right)$$

is true because of Definition 2.7 and

$$\pi_I \circ h ([\alpha_i]) = h (\pi_I \circ [\alpha_i]).$$

To see that this second equality is true, we analyse each side adding elements one by one. For the left hand side, start with

$$h ([\alpha_i]) : \tilde{F}_{hi} \rightarrow I \times G$$

$$F \mapsto (\alpha_I (h^{-1} F), \alpha_G (h^{-1} F)),$$

where $\alpha_I, \alpha_G$ are the components of $\alpha$. Adding the projection will yield the first component, that is

$$\pi_I \circ h ([\alpha_i]) : \tilde{F}_{hi} \rightarrow I$$

$$F \mapsto \alpha_I (h^{-1} F).$$

For the right hand side, we first need to project

$$\pi_I \circ [\alpha_i] : \tilde{F}_i \rightarrow I$$

$$F \mapsto \alpha_I (F),$$

and then add $h$

$$h (\pi_I \circ [\alpha_i]) : \tilde{F}_{hi} \rightarrow I$$

$$F \mapsto \alpha_I (h^{-1} F),$$

which proves the expression and thus the first item.

ii) With a proof analogous to the previous point,

$$\pi_G \circ h ([\alpha_i]) = h (\pi_G \circ [\alpha_i]).$$

iii) The following is true:

$$(g^{-1}z)_{hi} = h \left( (gh)^{-1}z \right)_{i}.$$

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Let us take the left hand side, which is
\[
\left( (g^{-1}z) \right)_{hi} = \left( (hh^{-1}g^{-1}z) \right)_{hi} = \left( h \left( (gh)^{-1}z \right) \right)_{hi}.
\]

Let \( \varphi = (gh)^{-1}z : \widetilde{F} \to G \). We only need to show that
\[
\left( h \varphi \right)_{hi} = h \left( \varphi_{hi} \right).
\]
The right hand side is clearly
\[
h \left( \varphi_{hi} \right) : \widetilde{F}_{hi} \to G
\]
\[
F \mapsto \varphi(h^{-1}F).
\]
For the left hand side, write first \( h \varphi \), which is
\[
h \varphi : \widetilde{F} \to G
\]
\[
F \mapsto \varphi(h^{-1}F),
\]
which under the restriction is
\[
\left( h \varphi \right)_{hi} : \widetilde{F}_{hi} \to G
\]
\[
F \mapsto \varphi(h^{-1}F),
\]
thus showing the last equivalence to get the item.

iv) Because the group action on \( G \) is free,
\[
h \left( \pi \circ \alpha_{hi} \right) = h \left( \left( (gh)^{-1}z \right)_{hi} \right) \iff \pi \circ \alpha_{hi} = \left( (gh)^{-1}z \right)_{hi}.
\]

Using i) and iv), we can rewrite the local tensors on \( hi \) as
\[
V_{hi}^{\varphi} = \begin{cases} V_{\varphi}^{\left( gh \right) \circ \alpha_{hi}} & \text{if } \pi \circ \alpha_{hi} = \left( (gh)^{-1}z \right)_{hi} \\ 0 & \text{else} \end{cases}
\]
by setting \( g' = gh \). Since the element \( g \) in the definition is not fixed beforehand, this is equivalent to the definition of \( V_{\alpha_{hi}}^{\varphi} \), thus showing that
\[
V_{hi}^{\varphi} = V_{\alpha_{hi}}^{\varphi},
\]
that is the second condition in Definition 2.10.

We now need to check the first condition, i.e. that the local tensors are an \( (\Omega, G) \) decomposition of \( v \). We start by identifying the nonzero terms of the sum. Rewrite \( \pi \circ \alpha_{hi} \) as maps \( \psi \) as in the following definition:
\[
\Psi = \{ \psi \in G^{\tilde{F}} \text{ such that } \forall i \exists g_i : \psi_{hi} = (g_i^{-1}z)_{hi} \}.
\]
Ψ is the set of maps that fulfil the condition for \( V^{[i]}_\alpha \) to be nonzero. Let \( i, j \in [n] \) be neighbouring points, and fix \( F \in \mathcal{F}_i \cap \mathcal{F}_j \). Then \( \forall \psi \in \Psi \)

\[
(g_i^{-1} z)_{|i}(F) = \psi_{|i}(F) = \psi_{|j}(F) = (g_j^{-1} z)_{|j}(F).
\]

Thus \( g_i z (F) = g_j z (F) \) which implies that \( g_i = g_j \). Because \( \Omega \) is connected, for a fixed \( \psi \in \Psi \) this extends to

\[
g_i = g_j \quad \forall i \in [n],
\]

and then \( \Psi \) becomes

\[
\Psi = \{ \psi \in G^{\mathcal{F}} \text{ such that } \exists g_i : \forall i \psi_{|i} = (g_{\overline{i}}^{-1} z)_{|i} \}.
\]

Additionally, let \( \pi_{\mathcal{I}} \circ \alpha = \beta \). With these last definitions, we obtain

\[
\sum_{\alpha \in \mathcal{I}^F} V^{[0]}_{\alpha_{|0}} \otimes \cdots \otimes V^{[n]}_{\alpha_{|n}} = \sum_{\psi \in \Psi} \sum_{\beta \in \mathcal{I}^F} W^{[g_0]}_{\alpha_{|0}}(\beta_{|0}) \otimes \cdots \otimes W^{[g_n]}_{\alpha_{|n}}(\beta_{|n}).
\]

For each fixed \( \psi \in \Psi \) the element of the sum is nothing else than \( g v = v \), because \( v \) is invariant. Thus

\[
\sum_{\alpha \in \mathcal{I}^F} V^{[0]}_{\alpha_{|0}} \otimes \cdots \otimes V^{[n]}_{\alpha_{|n}} = |\Psi| v \quad \Rightarrow \quad v = \sum_{\alpha \in \mathcal{I}^F} \frac{1}{|\Psi|} V^{[0]}_{\alpha_{|0}} \otimes \cdots \otimes V^{[n]}_{\alpha_{|n}}.
\]

By redefining on of the local tensors to absorb the constant \( |\Psi| \) we obtain an \((\Omega, G)\) decompositon of \( v \).

Thus, since \( |\mathcal{I} \times G| \) is finite,

\[
\text{rank}_{(\Omega, G)}(v) \leq |\mathcal{I} \times G| < \infty
\]

and the proof is finished.

Note that Theorem 2.2, together with Proposition 2.4, and Proposition 2.3 show that given a (nonweighted) simplicial complex with group action a tensor on its associated global vector space will have a finite invariant decomposition if and only if it is invariant under the group action. To achieve this, we need the weighted structure, since it allows us to convert any action into a free action. This is the main reason why we consider weighted simplicial complexes and not regular ones.

Finally we have another existance result, this time for blending group actions. Recall Definition 2.6, for the definition of blending group action.

**Theorem 2.3.** Let \((\Omega, G)\) be a connected wsc with group action. Let the action of \( G \) be blending on \([n]\). Then \( \forall v \in V^{\text{inv}} \) we have

\[
\text{rank}_{(\Omega, G)}(v) < \infty.
\]
Proof. Start with a finite decomposition
\[ v = \sum_{j \in \mathcal{I}} W_j^{[0]} \otimes \cdots \otimes W_j^{[n]} \]
of \( v \). Choose numbers \( d_i^{[l]} \) for \( i \in [n] \) and \( l = 1, \ldots, r \) (for some \( r \) large enough), such that
\[ \sum_{l=1}^{r} d_i^{[l]} \cdots d_i^{[n]} = \begin{cases} 1 & \text{if } \{i_0, \ldots, i_n\} = [n] \\ 0 & \text{else.} \end{cases} \]
This expression is a symmetric tensor rank decomposition of the tensor defined in the right hand side of the equation, which exists by [3]. Define the new local tensors as
\[ V_i^{[l]} = \begin{cases} \sum_{g \in G} d_i^{[g_0]} W_j^{[g_i]} & \text{if } \beta_i = j \in \mathcal{I} \text{ a constant function.} \\ 0 & \text{else,} \end{cases} \]
for \( i \in [n] \), \( l = 1, \ldots, r \), \( \beta \in \tilde{\mathcal{F}} \). For each fixed \( l \), condition 2 of Definition 2.10 is clearly fulfilled since the action of the group on itself is free. Moreover, since \( \Omega \) is connected
\[ \beta_i = j_1 \beta_k = j_2 \Rightarrow j_1 = j_2. \]
For each \( l \), we have that tensors \( V_i^{[l]} \) provide an \( (\Omega, G) \) decomposition of a certain \( v_l \in \mathcal{V} \). Now
\[ v_1 + \cdots + v_r = \sum_{l=1}^{r} \sum_{\alpha \in \tilde{\mathcal{F}}} V_i^{[0]} \otimes \cdots \otimes V_i^{[n]} \]
\[ = \sum_{g_0, \ldots, g_n \in G} \sum_{l=1}^{r} d_i^{[g_0]} \cdots d_i^{[g_n]} \sum_{j \in \mathcal{I}} W_j^{[g_0]} \otimes \cdots \otimes W_j^{[g_n]} \]
Where we have used that \( \Omega \) is connected to write a single \( j \) for all \( i \in [n] \). Now, since the group action is blending this can be written as:
\[ v_1 + \cdots + v_n = \sum_{g_0, \ldots, g_n \in G} \sum_{j \in \mathcal{I}} W_j^{[g_0]} \otimes \cdots \otimes W_j^{[g_n]} \]
\[ \propto \sum_{g \in G} \sum_{j \in \mathcal{I}} W_j^{[g]} \otimes \cdots \otimes W_j^{[g]} \]
\[ \sum_{g \in G} g \cdot v \]
\[ \propto v, \]
where \( \propto \) refers to a positive multiple. Now, since each \( v_l \) has rank \( |\mathcal{I}| \), from Proposition 2.2
\[ \text{rank}_{(\Omega,G)}(v) \leq r|\mathcal{I}| < \infty. \]
2.2.3 The invariant separable decompositon

The concept of separability is central in quantum information through its complementary: entanglement. In the following we define and study separable tensor decompositions. In the previous sections, $V_i$ could be any vectorspace. Now, we want decompositions that make evident certain structures of our tensors. Separability is a property of positive elements, so we need to consider vector spaces where separability is defined. We are interested on the space of multipartite density matrices. Thus, let each local vector space be

$$V_i = \mathcal{B}(\mathcal{H}_i),$$

$\forall i \in [n]$. Then the global vector space is

$$V = V_0 \otimes \cdots \otimes V_n = \mathcal{B}(\mathcal{H}_0) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n) \subseteq \mathcal{B}(\mathcal{H}_0 \otimes \cdots \otimes \mathcal{H}_n).$$

**Definition 2.12** (Invariant separable decomposition). Let $(\Omega, G)$ be a wsc with group action over $[n]$, $V$ as just defined and $\sigma \in V$. A separable $(\Omega, G)$ decomposition is an $(\Omega, G)$ decomposition of $\sigma$

$$\sigma = \sum_{\alpha \in \mathcal{I}} \sigma^{[0]}_{\alpha[0]} \otimes \cdots \otimes \sigma^{[n]}_{\alpha[n]}$$

such that

$$\sigma^{[i]}_{\alpha[i]} \succeq 0 \quad \forall i \in [n].$$

The minimum $|\mathcal{I}|$ such that this decomposition exists is the separable $(\Omega, G)$ rank, denoted by

$$\text{sep-rank}_{(\Omega, G)}(\sigma).$$

Note that in this framework, an $n$-partite element is separable if it admits a separable $(\Sigma_n, \{e\})$ decomposition. That is

$$\sigma = \sum_{j=0}^{r} \sigma^{[0]}_{j[0]} \otimes \cdots \otimes \sigma^{[n]}_{j[n]}.$$ 

Moreover, clearly a separable $(\Omega, G)$ decomposition can only exist if $\sigma$ is separable and $G$-invariant. The main result concerning the separable $(\Omega, G)$ decomposition states that all these elements have a finite separable $(\Omega, G)$ decomposition.

**Theorem 2.4.** Let $(\Omega, G)$ be a wsc with group action on $[n]$ such that $G$ acts freely on $\Omega$ and let $\sigma \in V_{\text{inv}}$ be separable. Then

$$\text{sep-rank}_{(\Omega, G)}(\sigma) \leq \infty.$$

---

Note that we are using the notation $\sigma$ for 2 different objects, the element to be decomposed and the factors of the decomposition. Since the factors always carry subindices this causes no ambiguities.
Proof. The proof is a direct consequence of the proof of Theorem 2.2 after taking an initial decomposition of psd elements, which can be done because $\sigma$ is separable. Since the proof does not modify these elements, it only indexes them differently and adds zeros (which are psd), the proof constructs a separable $(\Omega, G)$ decomposition starting from psd elements.

This kind of proof for results concerning the separable $(\Omega, G)$ rank is fairly common, since usually proofs concerning the $(\Omega, G)$ rank do not modify the factors and can be used without further additions for the separable $(\Omega, G)$ rank. Another example is the following result, which was Proposition 2.2 for the $(\Omega, G)$ rank.

**Proposition 2.5.** Let $(\Omega, G)$ be a connected wsc with group action on $[n]$. Then for $\sigma, \rho \in \mathcal{V}$ we have

$$\text{sep-} \text{rank}_{(\Omega, G)} (\sigma + \rho) \leq \text{sep-} \text{rank}_{(\Omega, G)} (\sigma) + \text{sep-} \text{rank}_{(\Omega, G)} (\rho).$$

**Proof.** The proof follows from the proof of Proposition 2.2. The manipulation in the proof turns psd factors of $\sigma$ and $\rho$ into psd factors of $\sigma + \rho$, thus yielding the result.

Finally, we want to remark that we have only used that separability is a convex property. Therefore this decomposition can be defined in particular for any convex cone, we chose separability because it is interesting to quantum information.

### 2.2.4 The invariant purification

We are interested in decomposition of positive elements. The separable decomposition shows this positivity of the decomposed object explicitly but is only applicable to some positive elements, the separable ones, as seen by Theorem 2.4. We want a decomposition that explicitly shows positivity that works for all positive elements. This is the invariant purification:

**Definition 2.13 (Invariant purification).** Let $(\Omega, G)$ be a wsc with group action on $[n]$ and let $\rho \in \mathcal{V} = \mathcal{B}(\mathcal{H}_0) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n)$. Then the $(\Omega, G)$ purification consists of the following:

Let $\mathcal{H}'_i$ be ancillary Hilbert spaces for each site $i \in [n]$. That is consider

$$\xi \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}'_0) \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n, \mathcal{H}'_n)$$

such that

$$\rho = \xi^\dagger \xi \text{ and } \text{rank}_{(\Omega, G)} (\xi) < \infty.$$

The minimum $\text{rank}_{(\Omega, G)} (\xi)$ such that this decomposition exists is called the $(\Omega, G)$ purification rank and is denoted by

$$\text{puri-} \text{rank}_{(\Omega, G)} (\rho).$$

Clearly this can only exists for positive elements, since anything written as a Hermitian square is positive. Moreover, the $\xi^\dagger$ is $G$-invariant if and only if $\xi$ is invariant, so that this decomposition can only exist for $G$-invariant elements.

Again we can replicate the results of Theorem 2.2 and Proposition 2.2 for the $(\Omega, G)$ purification.
**Theorem 2.5.** Let \((\Omega, G)\) be a wsc with group action on \([n]\) such that \(\text{rank}_{(\Omega, G)}(\xi) < \infty\) for all \(\xi \in \mathcal{V}_{\text{inv}}\). Also assume all \(\mathcal{H}_i\) are finite dimensional. Then for all psd \(\rho \in \mathcal{V}_{\text{inv}}\)

\[
\text{puri-rank}_{(\Omega, G)}(\rho) < \infty.
\]

**Proof.** If every \(\mathcal{H}_i\) then \(\mathcal{V} = \mathcal{B}(\mathcal{H}_0 \otimes \cdots \otimes \mathcal{H}_n)\) and every positive element \(\rho\) has a unique psd squareroot \(\xi \in \mathcal{V}\). \(\xi\) is a polynomial expression of \(\rho\), therefore it is also \(G\) invariant. Then, by assumption, it has a finite \((\Omega, G)\) decomposition. Thus the purification of \(\rho\) is finite. \(\square\)

Note that the hypothesis of the theorem are fulfilled if \(G\) is free or blending on \(\Omega\).

**Proposition 2.6.** Let \((\Omega, G)\) be a connected wsc with group action on \([n]\). Then for \(\rho, \sigma \in \mathcal{V}\)

\[
\text{puri-rank}_{(\Omega, G)}(\rho + \sigma) \leq \text{puri-rank}_{(\Omega, G)}(\sigma) + \text{puri-rank}_{(\Omega, G)}(\rho).
\]

**Proof.** We can assume that both \(\rho\) and \(\sigma\) admit finite \((\Omega, G)\) purifications (see the proof of Proposition 2.2), then

\[
\xi \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_0') \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n, \mathcal{H}_n')
\]

\[
\chi \in \mathcal{B}(\mathcal{H}_0, \mathcal{H}_0'') \otimes \cdots \otimes \mathcal{B}(\mathcal{H}_n, \mathcal{H}_n''),
\]

where

\[
\rho = \xi^\dagger \xi \quad \sigma = \chi^\dagger \chi.
\]

Consider both these elements as part of

\[
\mathcal{B}(\mathcal{H}_0, \mathcal{H}_0' \times \mathcal{H}_0'') \otimes \cdots \otimes (\mathcal{H}_n, \mathcal{H}_n' \times \mathcal{H}_n''),
\]

where each local operator maps to the respective first or second component. Then

\[
(\xi + \chi)^\dagger (\xi + \chi) = \xi^\dagger \xi + \chi^\dagger \chi = \rho + \sigma,
\]

and we can apply Proposition 2.2. \(\square\)

### 2.3 Characterisation of the \((\Omega, G)\) ranks

In this section we show some relations between the described ranks. Unless otherwise stated, fix a connected wsc with group action \((\Omega, G)\) on \([n]\) and let \(\mathcal{V}_i = \mathcal{B}(\mathcal{H}_i) \forall i \in [n]\) as in the previous sections. We start by comparing the different ranks:

**Proposition 2.7.** Let \(\rho \in \mathcal{V}\), then:

1) \(\text{rank}_{(\Omega, G)}(\rho) \leq \text{sep-rank}_{(\Omega, G)}(\rho)\)

2) \(\text{puri-rank}_{(\Omega, G)}(\rho) \leq \text{sep-rank}_{(\Omega, G)}(\rho)\)

3) \(\text{rank}_{(\Omega, G)}(\rho) \leq \text{puri-rank}_{(\Omega, G)}(\rho)^2\).
Proof. i) It is clear since the separable \((\Omega, G)\) decomposition is a special case (where each local tensor is psd) of the \((\Omega, G)\) decomposition.

For ii), let
\[
\rho = \sum_{\alpha \in \mathcal{I}} \sigma^{[0]}_{\alpha_{[0]}} \otimes \cdots \otimes \sigma^{[n]}_{\alpha_{[n]}}
\]
be an optimal separable \((\Omega, G)\) decomposition. Then let
\[
\tau^{[i]}_{\alpha_{[i]}} = \sqrt{\sigma^{[i]}_{\alpha_{[i]}}}
\]
and consider the bounded operator
\[
\xi^{[i]}_{\alpha_{[i]}}, : \mathcal{H}_i \longrightarrow \mathcal{H}_i' = \mathcal{H}_i \times \cdots \times \mathcal{H}_i
\]
where the product runs over \(\mathcal{I}\) and \(\tau^{[i]}_{\alpha_{[i]}} h\) appears on the entry indexed by \(\alpha_{[i]}\). These operators fulfill the second condition of Definition 2.10 and, thus, form an \((\Omega, G)\) decomposition of some \(\xi\), such that \(\text{rank}_{(\Omega, G)} (\xi) \leq |\mathcal{I}|\). By construction,
\[
\left(\xi^{[i]}_{\alpha_{[i]}}\right)^\dagger \xi^{[i]}_{\alpha_{[i]}} = \delta_{\alpha_{[i]}^{\prime} \alpha_{[i]} \sigma^{[i]}_{\alpha_{[i]}}}.
\]
Therefore \(\xi \dagger \xi = \rho\) and
\[
\text{puri-rank}_{(\Omega, G)} (\rho) \leq |\mathcal{I}| = \text{sep-rank}_{(\Omega, G)} (\rho).
\]

To show iii), assume \(\xi\) is an optimal purification of \(\rho\), then
\[
\text{rank}_{(\Omega, G)} (\rho) = \text{rank}_{(\Omega, G)} (\xi \dagger \xi) \leq \text{rank}_{(\Omega, G)} (\xi \dagger) \text{rank}_{(\Omega, G)} (\xi) = \text{rank}_{(\Omega, G)} (\xi)^2,
\]
which is the result. Note that we used Proposition 2.2 in the calculation.

2.3.1 Separation

For our purposes, studying decompositions of quantum states, the \((\Omega, G)\) rank seems useless, since it does not capture the fact that a quantum state is positive semidefinite, making it harder to implement in applied work. In this section we explain why we still study this rank. We want to compare the \((\Omega, G)\) rank with the \((\Omega, G)\) purification. Proposition 2.7 gives the following relation:
\[
\text{rank}_{(\Omega, G)} (\rho) \leq \text{puri-rank}_{(\Omega, G)} (\rho)^2.
\]

A natural question would be whether we can find an inverted relation, that is if there exists a function \(f\) such that
\[
\text{puri-rank}_{(\Omega, G)} (\rho) \leq f (\text{rank}_{(\Omega, G)} (\rho)) \quad \forall \rho.
\]
It was shown [7] that it is impossible to find such \(f\), even for \((\Lambda_1, \{e\})\).
Thus, the scalability of the \((\Omega, G)\) purification is much worse than that of the \((\Omega, G)\) decomposition. The relation of these two is summarised in Table 2.2.

<table>
<thead>
<tr>
<th>Decomposition</th>
<th>Positivity</th>
<th>Scalability</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\Omega, G)) decomposition</td>
<td>Unclear</td>
<td>Good</td>
</tr>
<tr>
<td>((\Omega, G)) purification</td>
<td>Clear</td>
<td>Bad</td>
</tr>
</tbody>
</table>

Table 2.2: Comparison of \((\Omega, G)\) purification and decomposition.

2.4 Correspondance to nonnegative tensors

There is a connection between the ranks of Section 2.2 and previously defined and useful ranks of nonnegative matrices, and an extension to general \((\Omega, G)\) ranks of nonnegative tensors. First we need a map that links these 2 spaces. In the case of matrices \((n=1)\), an inclusion can be written elementwise as follows. Assume \(d_0 = d_1 = d\) for simplicity and consider a diagonal element \(\rho\) of \(B(\mathcal{H}_0) \otimes B(\mathcal{H}_1)\), that is

\[
\rho = \sum_{i,j=1}^{d} \rho_{ij} |i, j\rangle |i, j\rangle ,
\]

with \(\rho_{ij} \geq 0\). Its associated element in \(\mathbb{C}^d \otimes \mathbb{C}^d = \mathcal{M}_d\) is

\[
M_{\rho} = \sum_{i,j=1}^{d} \rho_{ij} |i\rangle |j\rangle .
\]

This construction can be extended to the multipartite case as

\[
\rho = \sum_{i_0,\ldots,i_n=1}^{d} \rho_{i_0\ldots i_n} |i_0,\ldots,i_n\rangle |i_0,\ldots,i_n\rangle \quad \mapsto \quad M_{\rho} = \sum_{i_0,\ldots,i_n=1}^{d} \rho_{i_0\ldots i_n} |i_0,\ldots,i_n\rangle .
\]

In the following we give a general description of ranks of nonnegative tensors as well as how the \((\Lambda_1, \{e\})\) and \((\Lambda_1, \{\mathbb{Z}/2\mathbb{Z}\})\) ranks of \(\rho\) are analogous to previously studied ranks of \(M_{\rho}\). Note that the equivalence is bijective, to each diagonal psd matrix corresponds a unique normalised nonnegative tensor and viceversa.

It is this analogy that allows us to call this the classical case, since it can be seen as the case when we have a classical \((i.e.\ diagonal in the computational basis)\) state. The bipartite classical case consists of the following definitions:

\[\mathcal{M}_d = \mathcal{M}_d(\mathbb{C})\]
Definition 2.14.  

i) The usual rank. A minimal decomposition consists of matrices $P, Q \in \mathcal{M}_{d \times r}$ such that $M = PQ^T$. The rank of $M$, denoted by $\text{rank } M$, is the minimum $r$ such that this decomposition exists.

ii) The nonnegative rank [21, 10]. A nonnegative rank decomposition consists of matrices $P, Q \in \mathcal{M}_{d \times r}(\mathbb{R}_+)$ such that $M = PQ^T$. The nonnegative rank of $M$, denoted by $\text{rank}_+(M)$, is the minimum $r$ such that this decomposition exists.

iii) The psd rank [9]. A psd decomposition consists of psd matrices $E_i, F_j \in \mathcal{M}_r$ for $i, j = 1, \ldots, d$ such that

$$M = \sum_{i,j=1}^d \text{tr}\{E_i F_j^T\} \ket{i}angle\langle j|.$$

The psd rank of $M$, denoted by $\text{psd-rank}(M)$, is the minimum $r$ such that this decomposition exists.

And their symmetric versions:

iv) The symmetric rank. A symmetric rank decomposition consists of a matrix $P \in \mathcal{M}_{d \times r}$ such that $M = PP^T$. The symmetric rank of $M$, denoted by $\text{sym-rank}(M)$, is the minimum $r$ such that this decomposition exists.

v) The completely positive (cp) rank. A completely positive decomposition consists of a nonnegative matrix $P \in \mathcal{M}_{d \times r}(\mathbb{R}_+)$ such that $M = PP^T$. The cp rank of $M$, denoted by $\text{cp-rank}(M)$, is the minimum $r$ such that this decomposition exists.

vi) The completely positive semidefinite transposed (cpsdt) rank. A cpsdt decomposition consists of psd matrices $E_i \in \mathcal{M}_{d \times r}(\mathbb{R}_+)$ such that

$$M = \sum_{i,j=1}^d \text{tr}\{E_i E_j^T\} \ket{i}angle\langle j|.$$

The cpsdt rank of $M$, denoted by $\text{cpsdt-rank}(M)$, is the minimum $r$ such that this decomposition exists.

These ranks have been studied in many contexts (see for example [21, 10, 9]). Here we generalise these definitions to a multipartite case and then show its relation with the $(\Omega, G)$ decompositions of Section 2.2. First, we present multipartite classical ranks. Note that for simplicity we assume all spaces have the same dimension, which we denote by $d$.

Definition 2.15 (Classical ranks). Let $(\Omega, G)$ be a wsc with group action on $[n]$ and $M \in \mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d$ be a nonnegative $n + 1$-tensor. Then we have:

i) The $(\Omega, G)$ rank. It follows the same definition as in Definition 2.10.
ii) The nonnegative \((\Omega, G)\) rank. A nonnegative \((\Omega, G)\) decomposition is an \((\Omega, G)\) decomposition such that all local vectors are entrywise nonnegative. The associated rank is called nonnegative \((\Omega, G)\) rank and is denoted by \(\text{nn-rank}_{(\Omega,G)}(M)\).

iii) The psd \((\Omega, G)\) rank. A psd \((\Omega, G)\) decomposition consists of psd matrices

\[
E^{|i|}_j \in \mathcal{M}_{\mathcal{I}^g_i},
\]

for \(i \in [n], j = 1, \ldots, d\) such that

\[
\left( E^{|g|}_j \right)_{g_{\alpha_i}, g_{\alpha'_i}} = \left( E^{|i|}_j \right)_{\alpha_i, \alpha'_i},
\]

for all \(i \in [n], j = 1, \ldots, d, \alpha, \alpha' \in \mathcal{I}^g\) and \(g \in G\), and

\[
M = \sum_{j_0, \ldots, j_n = 1}^d \left( \sum_{\alpha, \alpha' \in \mathcal{I}^g} \left( E^{\alpha}_{j_0} \right)_{\alpha_{j_0}, \alpha'_{j_0}} \cdots \left( E^{\alpha}_{j_n} \right)_{\alpha_{j_n}, \alpha'_{j_n}} \right) |j_0, \ldots, j_n\rangle.
\]

The psd \((\Omega, G)\) rank, denoted by \(\text{psd-rank}_{(\Omega,G)}(M)\), is the minimum cardinality of \(\mathcal{I}\) such that this decomposition exists.

Note that the condition of \(M\) being nonnegative is not necessary for the first definition, but it is for the existence of the other two (clearly if each factor is nonnegative, the resulting element is nonnegative, for example).

The well known cases correspond to \(n = 1, \Omega = \Lambda_1\) and the 2 possible group actions, the trivial one and \(\mathbb{Z}/2\mathbb{Z}\), see Corollary 2.1 for the full equivalence. The following Theorem characterises the equivalence for all \((\Omega, G)\).

**Theorem 2.6.** Let \(\rho\) be a \(n + 1\)-partite psd matrix and \(M_\rho\) its associated nonnegative tensor. Then:

i) \(\text{rank}_{(\Omega,G)}(M_\rho) = \text{rank}_{(\Omega,G)}(\rho)\)

ii) \(\text{nn-rank}_{(\Omega,G)}(M_\rho) = \text{sep-rank}_{(\Omega,G)}(\rho)\)

iii) \(\text{psd-rank}_{(\Omega,G)}(M_\rho) = \text{puri-rank}_{(\Omega,G)}(\rho)\).

**Proof.** For i), start with a decomposition of \(M_\rho\), call each factor \(v^{[i]}_{\alpha_i} \in \mathbb{C}^d\). Then

\[
w^{[i]}_{\alpha_i} = \text{diag} \left( v^{[i]}_{\alpha_i} \right) \in \mathcal{M}_d.
\]

It is easy to check that this is an \((\Omega, G)\) decomposition of \(\rho\). For the converse, let \(w^{[i]}_{\alpha_i}\) be such that

\[
\rho = \sum_{\alpha \in \mathcal{I}^g} w^{[\alpha_{j_0}]}_{\alpha_{j_0}} \otimes \cdots \otimes w^{[\alpha_{j_n}]}_{\alpha_{j_n}},
\]
then the diagonal elements are
\[ \rho_{j_0, \ldots, j_n, j_0, \ldots, j_n} = \langle j_0, \ldots, j_n | \sum_{\alpha \in \mathcal{F}} w_{\alpha_0}^{[0]} \otimes \cdots \otimes w_{\alpha_n}^{[n]} | j_0, \ldots, j_n \rangle \]
\[ = \sum_{\alpha \in \mathcal{F}} \langle j_0 | w_{\alpha_0}^{[0]} | j_0 \rangle \otimes \cdots \otimes \langle j_n | w_{\alpha_n}^{[n]} | j_n \rangle. \]

Thus defining \( \nu_{\alpha_i}^{[i]} \) as the vector of diagonal elements \( w_{\alpha_i}^{[i]} \) yields a decomposition of \( M_\rho \).

ii) follows the same proof, since diagonal nonnegative matrices are psd and psd matrices have a nonnegative diagonal.

For iii) consider first a psd \((\Omega, G)\) decomposition of \( M_\rho \). Write each matrix \( E_j^{[i]} \) as a Gram matrix,
\[ \left( E_j^{[i]} \right)_{\alpha_{i}, \alpha'_{i}} = \left( a_{j, \alpha_{i}}^{[i]} \right)^\dagger_{\alpha_{i}, \alpha'_{i}} a_{j, \alpha_{i}}^{[i]}, \]
which is always possible for psd matrices. Then \( a_{j, \alpha_{i}}^{[i]} \in \mathbb{C}^{\mathcal{T}_\mathcal{F}} \) and
\[ a_{j, \alpha_{i}}^{[g_{j}]} = a_{j, \alpha_{i}}^{[i]}. \]

Now set
\[ \tau_{\alpha_i}^{[i]} = \sum_{j=1}^{d} a_{j, \alpha_{i}}^{[i]} \otimes E_{jj}, \]
where \( E_{jj} \) is the matrix with zeroes everywhere except for a 1 in position \( jj \). This is an \((\Omega, G)\) decomposition of \( \xi \) such that
\[ \rho = \xi^{\dagger} \xi \]
. On the other hand let \( \xi \) be such that \( \rho = \xi^{\dagger} \xi \) and consider an \((\Omega, G)\) decomposition
\[ \xi = \sum_{\alpha \in \mathcal{F}} \tau_{\alpha_0}^{[0]} \otimes \cdots \otimes \tau_{\alpha_n}^{[n]}. \]

Then define
\[ \left( E_{j}^{[i]} \right)_{\alpha_{i}, \alpha'_{i}} = \left( \left( \tau_{\alpha_{i}}^{[i]} \right)^\dagger_{\alpha_{i}, \alpha'_{i}} \tau_{\alpha_{i}}^{[i]} \right)_{\alpha_{i}, \alpha'_{i}} \]
which can be checked to form a psd \((\Omega, G)\) decomposition of \( M_\rho \).

As a corollary from Theorem 2.6 we can write the bipartite case \((\Omega = \Lambda_1)\). As stated, this case is especially important because the classical ranks have been studied independently and found uses in a variety of areas on applied mathematics.

**Corollary 2.1.** Let \( \rho \) be a bipartite psd matrix and \( M_\rho \) its associated nonnegative tensor. Then:

i) \( \text{rank}(M_\rho) = \text{rank}_{(\Lambda_1, \{e\})}(M_\rho) = \text{rank}_{(\Lambda_1, \{e\})}(\rho) \)
\[ ii) \; \text{nn-rank}(M_\rho) = \text{nn-rank}_{(\Lambda_1,\{e\})}(M_\rho) = \text{sep-rank}_{(\Lambda_1,\{e\})}(\rho) \]

\[ iii) \; \text{psd-rank}(M_\rho) = \text{psd-rank}_{(\Lambda_1,\{e\})}(M_\rho) = \text{puri-rank}_{(\Lambda_1,\{e\})}(\rho) \]

\[ iv) \; \text{sym-rank}(M_\rho) = \text{sym-rank}_{(\Lambda_1,Z/2Z)}(M_\rho) = \text{rank}_{(\Lambda_1,Z/2Z)}(\rho) \]

\[ v) \; \text{cp-rank}(M_\rho) = \text{nn-rank}_{(\Lambda_1,Z/2Z)}(M_\rho) = \text{sep-rank}_{(\Lambda_1,Z/2Z)}(\rho) \]

\[ vi) \; \text{cpsd-rank}(M_\rho) = \text{psd-rank}_{(\Lambda_1,Z/2Z)}(M_\rho) = \text{puri-rank}_{(\Lambda_1,Z/2Z)}(\rho) \]

Note that Corollary 2.1 makes explicit that the \((\Omega, G)\) usual, nonnegative and psd ranks are a generalisation of the known usual, nonnegative and psd ranks (as well as their symmetric counterparts). Finally, we can use Theorem 2.6 to find the following characterisation of classical ranks:

**Proposition 2.8.** Let \((\Omega, G)\) be a wsc with group action over \([n] \) and let \(M\) be an invariant nonnegative \(n\)-tensor, then:

\[ i) \; \text{rank}_{(\Omega,G)}(M) \leq \text{nn-rank}_{(\Omega,G)}(M) \]

\[ ii) \; \text{psd-rank}_{(\Omega,G)}(M) \leq \text{nn-rank}_{(\Omega,G)}(M) \]

\[ iii) \; \text{rank}_{(\Omega,G)}(M) \leq \text{psd-rank}_{(\Omega,G)}(M)^2 \]

**Proof.** Let

\[ M = \sum_{i_0,\ldots,i_n=1}^{d} M_{i_0\ldots i_n} |i_0,\ldots,i_n\rangle \]

and consider its associated classical quantum state

\[ \rho = \sum_{i_0,\ldots,i_n=1}^{d} M_{i_0\ldots i_n} |i_0,\ldots,i_n\rangle \langle i_0,\ldots,i_n| \]

Then the result is immediate from Theorem 2.6 and Proposition 2.7. \(\Box\)
3. Computational complexity of tensor factorisations

In this chapter, we study the computational complexity of decision problems associated to the tensor decompositions defined in Section 2.2. Our objective is to use Corollary 2.1 to find lower bounds to the computational complexity of these problems (see Section 3.2.1) based on the known results for the problems of classical ranks, which are a particular case of our problems. These results are discussed in Section 3.2.3. This is preliminary work for the study of computational complexity of our problems, therefore most of the following are partial results.

In Section 3.1 we give a theoretical introduction to computational complexity, as well the definitions we worked with within the field, and in Section 3.2 we present our results.

3.1 Introduction to computational complexity

In this section we give a very short introduction of basic computational complexity theory, based on [14], as well as the main complexity classes needed for this work. Section 3.1.1, Section 3.1.2 and Section 3.1.3 contain the introduction to complexity theory, and in Section 3.1.4 we introduce the classes relevant to this work.

3.1.1 Problems and solutions

The main object of study in computational complexity are problems. A problem is defined by its input and question (or output). The following is a well known problem, the generalisation to the Seven Bridges of Königsberg problem solved by Euler in 1736:

<table>
<thead>
<tr>
<th>EULERIAN PATH</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
</tbody>
</table>

where an Eulerian path is a path on $G$ that visits each edge exactly once. This problem is what is known as a decision problem, where the output is Boolean (yes or no).

Note the difference between a problem and an instance of a problem. In the example, EULERIAN PATH is a problem, and particular graph (the complete graph for example) is an instance of this problem.
Now that we have defined a problem, we need to define a solution. A solution to a particular instance of a problem is not considered a solution to the whole problem, in the sense that if a factorisation of 10 is found, the problem of factorising numbers is not solved. We need a general method to find a solution regardless of the particular instance, an *algorithm*. An algorithm is, intuitively defined, a sequence of elementary computation steps that will produce the desired output if carried out. A well known algorithm to solve Eulerian Path is to count the number of adjacent edges each vertex of the input graph has. There exists an Eulerian path if and only if there are 2 or 0 vertices with an odd number of adjacent edges.

Finally, we need to address the complexity of an algorithm. Here we consider time complexity, but there is an analogous definition for space complexity. The complexity of an algorithm is the time (in number of elementary steps) needed to solve any instance. This is known as worst case complexity. Given an algorithm to solve a problem, there might be a very small set of instances of a problem that take a very long time to solve, while most of the instances are fast. Worst case complexity would consider as runtime of the algorithm the time it takes to solve the longer instances. There are other conceptions of complexity, but worst case is the most widely used.

Given a problem, its complexity is the minimum over the complexities of all algorithms that solve the problem. Note that given a problem and an algorithm that solves it, it is hard to tell whether or not there is a faster algorithm to solve the problem. Thus, the complexity of a known algorithm gives an upper bound to the complexity of the problem. To find a lower bound, it is necessary to show that any algorithm would take at least a certain amount of time, which is fundamentally harder to do.

**Scaling and complexity classes**

If we consider the Eulerian Path problem and the definition of complexity given, it might seem that all problems have absurdly high complexity, because we are allowed to choose bigger and bigger graphs. This is why the computational complexity of a problem is always studied as a function of the input’s size. In the case of Eulerian Path it is the number of vertices of the input graph.

This leads into the concept of a complexity class. Roughly speaking, a complexity class is a class of problems with similar scaling. For example the class \( \text{TIME}(f(n)) \) is the class of problems that scale with a function of order \( f(n) \), where \( n \) is the size of the input.

### 3.1.2 P, NP and the Polynomial Hierarchy

The most important complexity classes are \( P \) and \( NP \). \( P \) is the complexity class of decision problems that can be solved in polynomial time, *i.e.*

\[
P = \text{TIME}(\text{poly}(n)).
\]

\( NP \) is the class of decision problems solved in non-deterministic polynomial time. A better informal definition is that \( NP \) is the class of decision problems such that if
the output to an instance is yes, then there is a simple proof. By simple we mean in polynomial time; often the terms hard and easy are used to mean non-polynomial and polynomial. The central question in computer science is the conjecture that $P \neq NP$, which says that these two classes are different but has not been proven so far\footnote{This is known as the P versus NP problem, which asks whether $P = NP$ or not. $P \neq NP$ is the most widely accepted answer to the problem. Throughout the introduction we assume the conjecture to be true, i.e. $P \neq NP$, as most literature on computational complexity does.}. On a fundamental level, this says that checking solutions is easier than finding them.

An alternative, more precise, definition of $NP$ is the following:

**Definition 3.1.** Problem $A$ is in $NP$: $x$ is a yes instance of $A$ if and only if there exists a $w$ such that $(x,w)$ is a yes instance of $B$, where $B$ is a decision problem in $P$ regarding pairs $(x, w)$, and where $|w| = \text{poly}(|x|)$. Formally:

$$A(x) = \exists w : B(x, w),$$

where $B \in P$ and $|w| = \text{poly}(|x|)$.

This last definition of $NP$, albeit not intuitive, allows us to easily introduce $coNP$. In the formal language of Definition 3.1, $A$ is in $coNP$ if and only if

$$A(x) = \forall w : B(x, w),$$

where $B \in P$ and $|w| = \text{poly}(|x|)$. $P$ and $coNP$ are not contained in each other but they both contain $P$. $NP$ problems usually ask whether there is an object that fulfils a certain property, while $coNP$ problems ask whether a collection of objects fulfil a certain property. That ties to their definition, $NP$ problems are easy if we are given a solution to test for a property and $coNP$ problems are easy if we have a counterexample to check that it does not fulfil a property.

Note that the definition of $NP$ and $coNP$ depends on $P$. We can change the complexity of problem $B$ in the definition to be a different class. Take, for example the definition of $coNP$ by changing $P$ by $NP$. We obtain $A$ in this class if

$$A(x) = \forall w : B(x, w)$$

such that $B$ is in $NP$ and $|w| = \text{poly}(|x|)$. Now, since $B$ is in $NP$ we can rewrite this as

$$A(x) = \forall w \exists w' : C(x, w, w')$$

with $C \in P$, $|w| = \text{poly}(|x|)$ and $|w| = \text{poly}(|x|)$. This class is known as $\Pi_2P$. In general, we can concatenate $\forall$ and $\exists$ quantifiers to define the following classes:

**Definition 3.2.** Problem $A$ is in $\Pi_kP$: $A(x) = \forall w_1 \exists w_2 \ldots Q w_k : B(x, w_1, \ldots, w_k)$, where $B \in P$, $|w_i| = \text{poly}(|x|)$ for all $i$ and $Q$ is $\forall$ or $\exists$. 

\footnote{This is known as the P versus NP problem, which asks whether $P = NP$ or not. $P \neq NP$ is the most widely accepted answer to the problem. Throughout the introduction we assume the conjecture to be true, i.e. $P \neq NP$, as most literature on computational complexity does.}
In the previous definition the quantifiers need to alternate, because if we have an
expression of the form \( \exists w_1 \exists w_2 \) with \( |w_i| = \text{poly}(|x|) \) for all \( i \), this is equivalent to \( \exists w_3 \) with \( |w_3| = \text{poly}(|x|) \) for \( w_3 = (w_1, w_2) \). Similarly to Definition 3.2:

**Definition 3.3.** Problem \( A \) is in \( \Sigma_k \P \):

\[
A(x) = \exists w_1 \forall w_2 \ldots Q w_k : B(x, w_1, \ldots, w_k),
\]

where \( B \in \P \), \( |w_i| = \text{poly}(|x|) \) for all \( i \) and \( Q \) is \( \forall \) or \( \exists \).

Note that \( \P = \Sigma_0 \P = \Pi_0 \P \), \( \NP = \Sigma_1 \P \) and \( \coNP = \Pi_1 \P \). Finally, we introduce the polynomial hierarchy, \( \PH \), as the union of all these classes:

\[
\PH = \bigcup_{k=0}^{\infty} \Sigma_k \P = \bigcup_{k=0}^{\infty} \Pi_k \P.
\]

The polynomial hierarchy is depicted in Fig. 3.1.

![Figure 3.1: The polynomial hierarchy.](image)

For the purpose of classical computers, problems in any class beyond \( \P \) are considered hard.

**Completeness**

We have stated that \( \P \neq \NP \) is a very important conjecture in computer science, but how could one attempt to solve it? A useful concept is that of completeness. A class complete problem is a problem that is not easier\(^2\) than any problem in the class, while still being in the class. For \( \NP \), a very well known example is the travelling salesman problem, among many others.

Another way to define a complexity class is through completeness. Let \( A \) be a problem, then \( \text{TIME}(A) \) is the class of problems that are not harder than \( A \).

\(^2\)See Section 3.1.3 for how to compare complexity.
3.1.3 Comparing problems: reductions

Let \( A \) and \( B \) be two problems. The \emph{Karp reduction} allows us to compare this two problems without finding individual algorithms and calculating a time for each.

**Definition 3.4** (Karp reduction). Let \( A, B \) be problems and denote by \( S_A, S_B \) the sets of instances of each problem. A Karp reduction from \( A \) to \( B \) is a map

\[
\varphi : S_A \rightarrow S_B
\]

such that for each instance \( x \in S_A \):

\[
B(\varphi(x)) = \text{yes} \iff A(x) = \text{yes}
\]

and \( |\varphi| = \text{poly}(|x|) \).

Fig. 3.2 shows a picture of sets \( S_A \) and \( S_B \) and the function \( f \). If there is a reduction from \( A \) to \( B \) we write that

\[
A \preceq_P B.
\]

Clearly, if \( B \) is in some class and \( A \preceq_P B \), then \( A \) is in said class as well.

![Figure 3.2: Scheme of a Karp reduction. Orange represents yes instances and green no instances. \( f \) must be polynomial in size.](image)

3.1.4 Existential theories

Finally, we need to introduce the classes that will be important in this work. These classes are known as existential theories, and can be thought of as the class of problems that are no harder than finding solutions of a polynomial in a certain field. See [16] for more information on existential theories. The definition formalises to:
Definition 3.5. Let $F$ be a field. The existential theory of $F$, denoted by $∃F$ is the class of problems of the form

$$∃x_1, \ldots, x_n \in F : P(x_1, \ldots, x_n),$$

where $P(x_1, \ldots, x_n)$ is a quantifier free formula or equalities and inequalities of polynomials with coefficients in $F$.

There is an existential theory class defined for each field, but we are only interested in the fields $\mathbb{R}$ and $\mathbb{C}$. Their complexities are situated as follows [2]:

$$\text{NP} \subseteq ∃\mathbb{C} \subseteq ∃\mathbb{R} \subseteq \text{PSPACE},$$

where $\text{PSPACE}$ is the class of problems that can be solved using a polynomial amount of space. Moreover [12],

$$∃\mathbb{C} \subseteq Σ_2^P,$$

while there is no similar result for $∃\mathbb{R}$. This leads us to believe that there might be a huge gap in complexity between $∃\mathbb{C}$ and $∃\mathbb{R}$, but lower bounds are in general really hard to find.

### 3.2 Complexity of rank problems

In this section we state the main results regarding the computational complexity of the problems introduced in Section 3.2.1. We would like to have, for every problem, a lower and upper bound to the computational complexity, and we want this bound to be the same. We present upper bounds to the complexity of our problems, as well as two lower bounds.

#### 3.2.1 Definition of rank problems

First, we introduce the main problems we are concerned with. All the problems we consider are of the following form: given an element and a natural number, is some rank of this element less or equal to the given natural number? Note that this formulation is a decision problem. That is because we do not concern ourselves with finding the decomposition, only the rank. In principle, it could be possible to solve this problem without having any information on the optimal decomposition that gives the rank. First, we need to define the state space, that is where we take our elements from.

**Definition 3.6.** We consider quantum states over the field $\mathbb{Q}(i)$. Let

$$\text{PSD}_d^n(\mathbb{Q}(i)) := \left( \bigotimes_{l=1}^n \mathcal{M}_d(\mathbb{Q}(i)) \right)^+ \cap \{ \rho \in \mathcal{M}_d(\mathbb{C}) \text{ s.t. } \text{Tr}\{\rho\} = 1 \}$$

be the set of quantum states with coefficients on $\mathbb{Q}(i)$ for $n$ quantum systems of dimension $d$. The set of separable density matrices over $\mathbb{Q}(i)$ in the same setting is

$$\text{SEP}_d^n(\mathbb{Q}(i)) := \text{conv}\{ \rho \in \text{PSD}_d^n(\mathbb{Q}(i)) | ∃\rho_l \in \text{PSD}_d(\mathbb{C}) \text{ with } \rho = \rho_1 \otimes \cdots \otimes \rho_n \}.$$
Remark 3.1. A typical set of quantum states is

\[ \{ \rho \in \mathcal{M}_d(\mathbb{C}) \text{ such that } \text{Tr}\{\rho\} = 1 \} \].

We use \( \mathbb{Q}(i) \) because we need the instance of our problem to have a finite expression, and irrational numbers do not always have such an expression. One could wonder why not add other irrational, but algebraic, numbers such as \( \sqrt{2} \). For a finite collection of algebraic numbers, the elements of \( \mathbb{Q}(i,A) \) have a finite expression. We consider only \( \mathbb{Q}(i) \) because this set is dense in \( \mathbb{C} \), thus for any state in \( \mathcal{M}_d(\mathbb{C}) \) there is an arbitrarily close approximation in \( \mathcal{M}(\mathbb{Q}(i)) \).

Now we can introduce the rank problems:

<table>
<thead>
<tr>
<th>(( \Omega, G )) RANK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
</tbody>
</table>

Note that for this problem \( \rho \) could be in any multipartite vector space. However, we are interested in decomposition of quantum states and will consider it as a problem over matrix spaces. Similarly, here we consider each subspace to be of the same dimension, which is not necessary in the general definition, but we do it for simplicity.

<table>
<thead>
<tr>
<th>sep-(( \Omega, G )) RANK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>puri-(( \Omega, G )) RANK</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Input:</strong></td>
</tr>
<tr>
<td><strong>Output:</strong></td>
</tr>
</tbody>
</table>

3.2.2 The separable rank problem

We defined sep-(\( \Omega, g \)) RANK as a problem with general quantum states as input. We now ask whether this choice of input space matters, i.e. if choosing the states to be separable makes the problem easier. Since knowing whether a quantum state is separable or not is a hard\(^3\) problem, we would expect this choice to matter. The fact that this matters is also why we define it with input on all quantum states, since starting from a separable state is not trivial. First, consider the following 2 sep-(\( \Omega, g \)) RANK problems, throughout this section we refer to them as A and B.

A. Same as sep-(\( \Omega, g \)) RANK, that is \( \rho \in \text{PSD}_{d^n}(\mathbb{Q}(i)) \).

B. sep-(\( \Omega, g \)) RANK but with \( \sigma \in \text{SEP}_{d^n}(\mathbb{Q}(i)) \) as input.

\(^3\)i.e. not in P
In this section we show that A is indeed strictly harder than B. First the following
lemma is shown:

**Lemma 3.1.**  
1) $\text{sep-rank}_{(\Omega,G)}(\sigma) \leq |G|d^{2n(n+1)} \forall \sigma \in \text{SEP}_{d^n}(Q(i))$.
2) $B \preceq_P A$.

**Proof.**  
1) Concatenate [5, Proposition 31], [5, Proposition 36], and [6, Proposition 49 i)].
2) Consider 

$$i : \text{SEP}_{d^n}(Q(i)) \rightarrow \text{PSD}_{d^n}(Q(i))$$

the inclusion function from the set of separable states to the set of quantum states. This function is polynomial and is a Karp reduction from B to A, since the separable $(\Omega, G)$ rank of a separable state $\sigma$ is an intrinsic property of this state.

We can now state the main Theorem of this section:

**Theorem 3.1.** A is strictly harder than B, i.e.

$$A \not\preceq_P B.$$  

**Proof.** Let us assume that $A \preceq_P B$. Then 

$$\forall r \in \mathbb{N} \ \exists f_r : \text{PSD}_{d^n}(Q(i)) \rightarrow \text{SEP}_{d^n}(Q(i))$$

polynomial and such that 

$$A(\rho, r) = B(f(\rho), r) \forall \rho \in \text{PSD}_{d^n}(Q(i)).$$

Let $\rho_0$ be a non-separable quantum state, i.e. $\rho_0 \in \text{PSD}_{d^n}(Q(i))$ but $\rho_0 \not\in \text{SEP}_{d^n}(Q(i))$. By definition $\text{sep-rank}_{(\Omega,G)}(\rho_0) = \infty$, therefore $A(\rho_0, r) = NO \forall r \in \mathbb{N}$.

Let $r_0 = |G|d^{2n(n+1)} + 1$. From Lemma 3.1 i), $\sigma \in \text{SEP}_{d^n}(Q(i)) B(\sigma, r_0) = YES$. Then 

$$A(\rho_0, r_0) = NO \land B(f_{r_0}(\rho_0), r_0) = YES,$$

which contradicts the definition of $f$. Thus we get the result.

It is easy to see the following corollary of the proof:

**Corollary 3.1.** Let $\text{sepTest}$ be the problem of of deciding whether an input quantum state is separable. Then 

$$A(\Omega = \Lambda_n, G = \{e\}, r = d^{n(n+1)}) \equiv_P \text{sepTest}.$$  

Note that $d^{n(n+1)} = r_0 - 1$ for the case $G = \{e\}.$

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3.2.3 Bounds

In this section we present the bounds for the defined problems. Starting with \((\Omega, G)\) RANK:

**Theorem 3.2.** \((\Omega, G)\) RANK is in \(\exists \mathbb{C}\).

**Proof.** Consider the input element of \((\Omega, G)\) RANK: \(n, r, d \in \mathbb{N}\) and \(\rho \in \text{PSD}_{d^n}(\mathbb{Q}(i))\). A rank \(r\) \((\Omega, G)\) decomposition of \(\rho\) is an index set \(I\) such that \(|I| = r\) and matrices \((v^{|l|}_\alpha)_{\alpha \in I}\) \(\in \mathcal{M}_d(\mathbb{C})\), such that

\[
\rho = \sum_{\alpha \in I} v^{|0|}_\alpha \otimes \cdots \otimes v^{|n|}_\alpha.
\]

Each \((v^{|l|}_\alpha)_{\alpha \in I}\) can be decomposed in a basis as

\[
v^{|l|}_\alpha = \sum_{i, j = 1}^{d} \left(v^{|l|}_\alpha\right)_{i, j} |i\rangle\langle j|.
\]

Thus the decomposition becomes

\[
\rho = \sum_{\alpha \in I} \sum_{i_0, \ldots, i_n = 1}^{n} \left(v^{|0|}_\alpha\right)_{i_0, j_0} \cdots \left(v^{|n|}_\alpha\right)_{i_n, j_n} |i_0, \ldots, i_n\rangle\langle j_0, \ldots, j_n|.
\]

By taking each component of \(\rho\) with the same basis we obtain the following \(d^{2n+2}\) equations:

\[
(\rho)_{i_0, \ldots, i_n, j_0, \ldots, j_n} = \sum_{\alpha \in I} \left(v^{|0|}_\alpha\right)_{i_0, j_0} \cdots \left(v^{|n|}_\alpha\right)_{i_n, j_n}.
\]

Moreover, we need to consider that for all \(l \in [n], g \in G\) and \(\alpha \in I\),

\[
v^{|l|}_\alpha = v^{|lg|}_{\alpha g}.
\]

This adds a maximum of \(n \cdot n!\) equations. In practice, all these do is reduce the number of variables, since all these equations are 1 to 1 equalities between variables of the previous equations.

Thus we have a set of at most \(d^{2n+2}\) polynomial equations with at most \(rd^{n+2}\) unknowns. That is we have a statement of the form

\[
\exists x_1, \ldots, x_{nd^2} \in \mathbb{C} \mid P(x_1, \ldots, x_{nd^2})
\]

with \(P(x_1, \ldots, x_{nd^2})\) a quantifier free formula involving equalities of complex polynomials.

The size of \(P(x_1, \ldots, x_{nd^2})\) is \(\sim rd^{2n+2}\), which is polynomial with the size of the input (the input grows exponentially in size with \(n\)), thus we can consider the input to be the dimension of \(\rho\), \(d^2\) and \(r\).
For SEP-$(\Omega, G)$ RANK and PURI-$(\Omega, G)$ RANK we need to check whether our matrices are psd, so this approach does not work (see Proposition 3.1). We show instead that they are in $\exists \mathbb{R}$. First the following Lemma is needed:

**Lemma 3.2.** Let $(\Omega, G)$ be a wsc of size $n$ with group action and $\rho \in \text{PSD}_{d^n}(\mathbb{Q}(i))$ invariant under $G$ and such that $\text{puri-rank}_{(\Omega, G)}(\rho) \leq r$. Then

$$s_l \leq dr|\tilde{F}_l|.$$

**Proof.** Since $\text{puri-rank}_{(\Omega, G)}(\rho) \leq r$, we can consider a size $r$ $(\Omega, G)$ purification of $\rho$, that is $\xi$ such that

$$\rho = \xi^\dagger \xi$$

and such that $\xi$ admits an invariant $(\Omega, G)$ decomposition of size $r$:

$$\xi = \sum_{\alpha \in \tilde{I}} v_{\alpha[0]}^{[l]} \otimes \ldots \otimes v_{\alpha[n]}^{[n]}$$

with $|\tilde{I}| = r$, $v_{\alpha[l]}^{[l]} \in M_{s_l \times s_l}(\mathbb{C})$. Then consider the element

$$A_{\alpha[l], \alpha'[l]}^{[l]} = v_{\alpha[l]}^{[l]} \ast v_{\alpha'[l]}^{[l]}.$$

Note that $A^{[l]}$ is a bounded operator over

$$\mathcal{H} = M_{d}(\mathbb{C}) \otimes \left( \bigotimes_{l} \mathcal{M}_{s_l}(\mathbb{C}) \right)$$

that is psd by construction. Thus it can be expressed as a square in $B(\mathcal{H})$. Therefore

$$s_l \leq \dim \mathcal{H} = dr|\tilde{F}_l|.
\]

This bound is polynomial for reasonable $\Omega$, since the number of facets per vertex, $|\tilde{F}_l|$, is independent of the number of vertices. Now we can state the result for PURI-$(\Omega, G)$ RANK:

**Theorem 3.3.** PURI-$(\Omega, G)$ RANK $\in \exists \mathbb{R}$.

**Proof.** Let $n, r, d \in \mathbb{N}$, a wsc $\Omega$ on $[n]$, a group $G$ with group action on $\Omega$ and a density matrix $\rho \in \text{PSD}_{d^n}(\mathbb{Q}(i))$ be the input of an instance of PURI-$(\Omega, G)$ RANK. Then $\rho = \xi \xi^\dagger$ with $\xi \in M_{d^n \times s^o}(\mathbb{C})$ and such that the coefficients fulfill $\xi_{ik} = \overline{\xi_{ki}}$, i.e.
\[ \xi^\dagger_{kj} = \xi^r_{jk} - i \xi^i_{jk}. \] Then the real part of the \( ij \)th component of \( \rho \) is

\[
\rho^r_{ij} = \Re \left\{ \sum_{k=1}^{s^n} \xi_{ik} \xi^\dagger_{kj} \right\} = \frac{1}{2} \sum_{k=1}^{s^n} \xi_{ik} \xi^\dagger_{kj} + \xi^\dagger_{ik} \xi_{kj} \\
= \sum_{k=1}^{s^n} (\xi^r_{ik} + i \xi^i_{ik}) (\xi^r_{jk} - i \xi^i_{jk}) + (\xi^r_{ik} - i \xi^i_{ik}) (\xi^r_{jk} + i \xi^i_{jk}) \\
= \frac{1}{2} \sum_{k=1}^{s^n} \xi^r_{ik} \xi^r_{jk} - i \xi^r_{ik} \xi^i_{jk} + \xi^i_{ik} \xi^r_{jk} + \xi^r_{ik} \xi^i_{jk} + i \xi^r_{ik} \xi^i_{jk} + i \xi^i_{ik} \xi^r_{jk} \\
= \sum_{k=1}^{s^n} \xi^r_{ik} \xi^r_{jk} + \xi^i_{ik} \xi^i_{jk},
\]

which is a polynomial equation over \( \mathbb{R} \).

Now we can consider an invariant decomposition of \( \xi \). That is \( v^l_j \in M_{d \times s}(\mathbb{C}) \), \( \forall l \in [n] \) and with \( \alpha \in \mathcal{I}^\tilde{F} \) such that \( \mathcal{I} \) is an index set fulfilling \( |\mathcal{I}| = r \) and \( \tilde{F} \) the facets of \( \Omega \) (see [5]). We can see \( i, k \) as multindices, i.e. \( i = (i_0, \ldots, i_n) \), \( k = (k_0, \ldots, k_n) \). The coefficients of \( \xi \) are then

\[
\xi_{ik} = \sum_{\alpha \in \mathcal{I}^\tilde{F}} (v^{[0]}_{\alpha})_{i_0,k_0} \cdots (v^{[n]}_{\alpha})_{i_n,k_n}.
\]

And the real part

\[
\xi^r_{ik} = \Re \{ \xi_{ik} \} = \frac{1}{2} \sum_{\alpha \in \mathcal{I}^\tilde{F}} (v^{[0]}_{\alpha})_{i_0,k_0} \cdots (v^{[n]}_{\alpha})_{i_n,k_n} + (v^{[0]}_{\alpha})^*_{i_0,k_0} \cdots (v^{[n]}_{\alpha})^*_{i_n,k_n} \\
= \frac{1}{2} \sum_{\alpha \in \mathcal{I}^\tilde{F}} \left( (v^{[0]}_{\alpha})^r_{i_0,k_0} + i (v^{[0]}_{\alpha})^i_{i_0,k_0} \right) \cdots \left( (v^{[n]}_{\alpha})^r_{i_n,k_n} + i (v^{[n]}_{\alpha})^i_{i_n,k_n} \right) \\
+ \left( (v^{[0]}_{\alpha})^r_{i_0,k_0} - i (v^{[0]}_{\alpha})^i_{i_0,k_0} \right) \cdots \left( (v^{[n]}_{\alpha})^r_{i_n,k_n} - i (v^{[n]}_{\alpha})^i_{i_n,k_n} \right)
\]

Now consider \( \beta \in \{ r, i \}^{|n|} \) and

\[
|\beta|_i = |\{ l \in [n] \text{ s.t. } \beta(l) = i \}|.
\]

Then the previous equation is

\[
\xi^r_{ik} = \frac{1}{2} \sum_{\alpha \in \mathcal{I}^\tilde{F}} \sum_{\beta \in \{ r, i \}^{|n|}} i^{\frac{|\beta|_i}{2}} (v^{[0]}_{\alpha})^{\beta_0}_{i_0,k_0} \cdots (v^{[n]}_{\alpha})^{\beta_n}_{i_n,k_n} \\
+ \left( -1 \right)^{|\beta|_i} i^{\frac{|\beta|_i}{2}} (v^{[0]}_{\alpha})^{\beta_0}_{i_0,k_0} \cdots (v^{[n]}_{\alpha})^{\beta_n}_{i_n,k_n} \\
= \sum_{\alpha \in \mathcal{I}^\tilde{F}} \sum_{\beta \in \{ r, i \}^{|n|} \text{ s.t. } |\beta|_i \geq 2} i^{\frac{|\beta|_i}{2}} (v^{[0]}_{\alpha})^{\beta_0}_{i_0,k_0} \cdots (v^{[n]}_{\alpha})^{\beta_n}_{i_n,k_n}.
\]

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Since in the last expression $|\beta_i|$ is a multiple of 2, $i|\beta_i| = \pm 1$ making this a real polynomial. A similar argument can be done for the imaginary part.

We can plug the expressions for $\xi_{ik}$ and $\xi_{ik}^*$ in the expressions for $\rho_{ij}$ and $\rho_{ij}^*$, obtaining $2d^{2(n+1)}$ degree $2(n+1)$ polynomial equations with at most $2(n+1)dsr$ unknowns. Lemma 3.2 gives an upper bound for $s$ as a function of the input, allowing us the write the previous bound as a polynomial of the size of the input only.

**Theorem 3.4.** \( sep-(\Omega, G) \) \( \mathsf{rank} \in \exists \mathbb{R} \).

**Proof.** The calculation is equivalent to that of the previous Theorem. In this case clearly $s \leq d$; since each element $\sigma_{ij}^{[j]}$ is psd. Thus we can split it within the $C^*$-algebra $\mathcal{M}_d(\mathbb{C})$, meaning that the inner dimension $s$ is at most the space dimension $d$.

In total we get $2d^{2(n+1)}$ degree $2(n+1)$ polynomial equations over at most $2nd^2r$ variables.

Finally, we write why $\exists \mathbb{R}$ is an acceptable upper bound to the complexity of $sep-(\Omega, G) \ \mathsf{rank}$ and $puri-(\Omega, G) \ \mathsf{rank}$, that is why the proof of Theorem 3.2 cannot be adapted to Theorem 3.3 and Theorem 3.4. Consider the following:

**Proposition 3.1.** There is no polynomial $p(z) \in \mathbb{C}[z]$ such that $p(z) = z$.

**Proof.** If there exists such a $p(z)$, then necessarily

$$p(p(z)) = p(\bar{z}) = \bar{z} = z.$$  

Suppose that $p$ has degree $n$, then it can be written as

$$p(z) = a_n z^n + \ldots a_1 z + a_0$$

with $a_n \neq 0$. Then

$$p(p(z)) = a_n (a_n z^n + \ldots a_1 z + a_0)^n + \cdots + a_0$$

$$= a_n \left( a_n^2 z^{2n} + q_{2n-1}(z) \right) + \cdots + a_0$$

$$= a_n^{n+1} z^{n^2} + q_{2n-1}(z) = z \ \forall z.$$  

Necessarily then $n = 1$, $a_1 = \pm 1$ and $a_0 = 0$, but $z \neq \bar{z} \neq -z$ in general. Thus $p(z)$ does not exist.

The proof of Theorem 3.3 shows that we need the complex conjugate of the elements of $\rho$ to be able to write the problem as a polynomial. Since we can not take this conjugate as polynomials over $\mathbb{C}$, it becomes necessary to consider polynomials on $\mathbb{R}$, thus obtaining $\exists \mathbb{R}$ as a bound.

Finally, we show some lower bounds, as well as a result for a slightly varied case of $puri-(\Omega, G) \ \mathsf{rank}$.

**Theorem 3.5.** i) \( \langle \Lambda_n, \{e\} \rangle \) \( \mathsf{rank} \) is in $P$
ii) **SEP-(Ω, G) RANK** is NP-hard.

**Proof.** [6] provides an algorithm to solve \((Λ_n, \{e\})\) RANK through consecutive SVD. This is a polynomial time algorithm. Therefore \((Λ_n, \{e\})\) RANK is in \(P\).

For ii), let us first introduce the following problem:

| **NN RANK** |
|-----------------|-----------------|
| **Input:**      | Numbers \(r, d \in \mathbb{N}\) and a nonnegative matrix \(M \in M_{d^2}(\mathbb{Q})\). |
| **Output:**     | Yes if \(\text{rank}_+(M) \leq r\) and no if \(\text{rank}_+(M) > r\). |

Then, **SEP-(Λ_1, G) RANK** is at least as hard as **NN RANK** from the equivalence in Corollary 2.1. **NN RANK** has a algorithm with time \(d^{O(r^2)}\) (for \(n = 2\)) [1, 13], and in particular is NP-hard [17, 18].

\(\square\)

There are no equivalent results for **PURI-(Ω, G) RANK**.
4. Conclusions

We have introduced a framework to encode physical structures for tensor decompositions, a weighted simplicial complex $\Omega$ for structure and a group $G$ acting on $\Omega$ for the external symmetries of the system. We have also introduced three different decompositions based on this structure. Since our aim is to use these decompositions in positive semidefinite matrices, we have introduced decompositions that explicitly encode convex properties. We have shown that decompositions are finite under freeness of the group action, and that every wsc with group action $(\Omega, G)$ can be modified to get this freeness condition, effectively making every invariant state in a system have a finite decomposition. Moreover, we have shown how the associated ranks behave under operations and how they relate to each other.

We have also shown how these concepts generalise well known decompositions of nonnegative matrices.

Finally, we have introduced decision problems related to the $(\Omega, G)$ decompositions and have shown some preliminary results in the form of upper bounds to the complexity of these problems. We are currently working on the completion of these results, which ideally would turn into each problem having the same upper and lower bound to the complexity.

Another route is to consider approximate tensor decompositions, that is a the minimum $r$ such that

$$v \simeq \sum_{\alpha \in \mathcal{I}^F} V_{\alpha_0}^{[0]} \otimes V_{\alpha_1}^{[1]} \otimes \cdots \otimes V_{\alpha_n}^{[n]},$$

where $\simeq$ means

$$\left| v - \sum_{\alpha \in \mathcal{I}^F} V_{\alpha_0}^{[0]} \otimes V_{\alpha_1}^{[1]} \otimes \cdots \otimes V_{\alpha_n}^{[n]} \right| < \varepsilon$$

for some $\varepsilon > 0$. It would be interesting to see how the results change with this new definition, since in physics it is not required (or possible) to work with exact states.
Bibliography


