THE STABILITY OF NON-COMMUTATIVE QUADRATIC MODULES

A thesis submitted for the degree of Doctor of Philosophy

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universität Innsbruck

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November 2019
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1 Introduction

A few decades ago non-commutative geometry emerged together with objects that can be viewed as non-commutative counterparts of already known objects in commutative algebra and other fields. For example, the Hochschild cohomology is the non-commutative version of the de Rham cohomology in the domain of non-commutative geometry. Further examples can be found in [14].

This motivates to investigate the non-commutative versions of so called quadratic modules in real algebra. We will use the notion of stability and total stability as a mean to compare a quadratic module with its non-commutative counterpart.

In this work we will investigate non-commutative quadratic modules with respect to the two mentioned properties. By doing so, we will extend the known theory in multiple ways and encounter phenomena that do not appear in the commutative case.

The main result contain the following:

- We will show that the total stability of a finitely generated quadratic module is almost characterized by a certain regular sequence. This is an example of a phenomenon that does not appear in the commutative case.

- We will give combinatorial conditions that are sufficient for a non-commutative quadratic module to be totally stable.

- We will introduce the so called hull-kernel resp. Jacobson topology and give a sufficient geometrical criterion for a non-commutative quadratic module to be stable.

- We introduce a new object called a matrix tentacle and we will investigate certain non-commutative semialgebraic sets with this new tool.

- The investigation of non-commutative semialgebraic sets is continued by using the notion of a joint spectrum. We present furthermore an application with respect to minimal and maximal operator systems.

- Finally, we turn our attention to the quadratic module of all the sums of square with respect to certain group algebras.
Acknowledgments: First of all I want to thank my supervisor Tim Netzer for giving me the possibility of writing this PhD thesis under his supervision and for being patient and tolerant with me and my work. I am very grateful for the degree of freedom under his supervision and enjoyed working in his group.

I also want to thank Igor Klep, not just for giving me an opportunity to have an extended stay at the university of Ljubljana, but also for interesting discussions.
In this chapter we want to introduce the notion of a tentacle. These objects, by their very definition, are a geometric way to study the behavior of polynomials for large values. A similar tool also exists in the study of toric varieties, where the one parameter group is used as a tool to study the orbits of certain actions on toric varieties. See for example chapter 3.2 in [17] for more details. As one parameter groups tell us something about certain geometric properties of toric varieties, tentacles tell us something about algebraic properties of objects called quadratic modules. The pioneer work with respect to that matter has been done in [39].

The aim of this chapter is to introduce the notion of a tentacle and other central objects in order to pave the way for future applications in the next chapters.

1.1 Preliminaries

Definition 1.1.1. Let \( V \) be a subset of \( \mathbb{R}^n \) with non-empty interior. A tentacle \( T_K \) of \( V \) consists of the following data

- A tuple \((\varphi_1, \ldots, \varphi_n)\) \( \in \mathbb{R}(t)^n \) which we call the defining functions of \( T_K \).
- A compact set \( K \subseteq V \) with non-empty interior which we call the basis of \( T_K \).

that is put together into the form

\[
T_{K,z} = \{(\varphi_1(t)x_1, \ldots, \varphi_n(t)x_n) : (x_1, \ldots, x_n) \in K, t \geq 1, \varphi_1(t), \ldots, \varphi_n(t) \text{ is defined}\},
\]

where \( z = (\deg(\varphi_1), \ldots, \deg(\varphi_n)) \). The set of all tentacles \( T \) that are contained in \( V \) is denoted with \( T(V) \). We will usually write \( T \) instead of \( T_K \). The element \( z \in \mathbb{Z}^n \) that is given by \( z = \deg(T) := (\deg(\varphi_1), \ldots, \deg(\varphi_n)) \) is called the degree of \( T \). The set \( T^+(V) \) is a subset of \( T(V) \) that consists of all \( T \in T(V) \) for which \( \deg(T) \) has at least one positive component.

Definition 1.1.2. Let \( V \) be a subset of \( \mathbb{R}^n \) with non-empty interior. A fiber \( \text{Fibre}_x(T) \) of \( T \) for some point \( x \in K \) is given by

\[
\text{Fibre}_x(T) = \{(\varphi_1(t)x_1, \ldots, \varphi_n(t)x_n) : t \geq 1, \varphi_1(t), \ldots, \varphi_n(t) \text{ is defined}\}.
\]

Remark 1.1.3. Here are some simple facts about tentacles:

- Let \( S \subseteq \mathbb{R}^n \) be a semialgebraic set with non-empty dense interior and let \( T \in T(S) \). Suppose that there is a fiber \( \text{Fibre}_x(T) \) of \( T \) such that \( \text{Fibre}_x(T) \cap \text{int}(S) \) is unbounded. Then \( \text{Fibre}_x(T) \) gives rise to an element \( T' \in T(S) \) in the following way: There is a \( t' \geq 1 \) such that the corresponding fiber elements are all contained in \( \text{int}(S) \). Using the fact that polynomials depend continuously on its coefficients, we can easily find a neighborhood \( U \) of \( x \) and a tentacle \( T' \) with \( U \) as its basis such that \( T' \in T(S) \).
Let \( T \in \mathbb{T}(\mathbb{R}^n) \) and \( f : \mathbb{R}^n \to \mathbb{R}^m \) be an isomorphism of vector spaces. Then \( f(T) \) contains an element out of \( \mathbb{T}(\mathbb{R}^n) \). Thus \( \mathbb{T}(f(T)) \neq \emptyset \).

Let \( T_1, \ldots, T_j \in \mathbb{T}(\mathbb{R}^n) \). Then \( T_1, \ldots, T_j \) give rise to an element \( T \in \mathbb{T}(\mathbb{R}^{jn}) \) in an obvious way: The basis of \( T \) is the direct product of the bases of the \( T_1, \ldots, T_j \) and the defining functions of \( T \) are given by the defining functions of \( T_1, \ldots, T_j \). We will say that \( T \) is the LIFT of the \( T_1, \ldots, T_j \).

**Remark 1.1.4.** Let \( T \in \mathbb{T}(\mathbb{R}^n) \) with defining functions \((\varphi_1, \ldots, \varphi_n) \in \mathbb{R}(t)^n \setminus \{0\} \). Take a covering \((U_i)_i\) of affine open sets of \( \mathbb{A}^1 \) such that \( \frac{\partial}{\partial t} \in H^0(U_i \cap U_j, \mathcal{O}_{\mathbb{A}^1}^\times) \) for \( i \neq j \). By [52, Definition, p. 153], this data gives rise to a CARTIER DIVISOR \( D \) on \( \mathbb{A}^1 \). Using [53, Theorem 6.1, p. 64] the Cartier divisor \( D \) corresponds up to equivalence to a LINE BUNDLE \( \mathcal{L}_D \). This correspondence motivates the definitions in Definition [1.1.1] and especially in Definition [1.1.2].

Let \( S \subseteq \mathbb{R}^n \) be a semialgebraic set. Suppose that \( T \in \mathbb{T}(S) \) has the property that \( z_1 = \cdots = z_n \) for \( z = \deg(T) \) and \( z_1 > 0 \). Let \( \mathbb{P}^n_\mathbb{R} \) be the \( n \)-dimensional weighted projective space with weight \((z_1, z)_1 \). Then \( T \in \mathbb{T}(S) \) implies that the closure \( \overline{S} \) of \( S \) in \( \mathbb{P}^n_\mathbb{R} \) defines a Zariski dense subset \( \mathbb{S} \cap (\mathbb{P}^n_\mathbb{R} \setminus \{0\}) \) of \( \mathbb{P}^n_\mathbb{R} \setminus \mathbb{A}^n \). Conversely, any Zariski dense subset of \( \mathbb{P}^n_\mathbb{R} \setminus \mathbb{A}^n \) in \((\mathbb{P}^n_\mathbb{R} \setminus \mathbb{A}^n)(\mathbb{R})\) gives rise to an element \( T \in \mathbb{T}(S) \) such that all components of \( z = \deg(T) \) are positive and equal. This is a special case of [44, Theorem 2.14, p. 80].

### 1.2 Tentacles and Geometry

**Definition 1.2.1.** Let \( V \) be a subset of \( \mathbb{R}^n \). We define \( c(V) \) to be the biggest natural number \( l \) such that there are tentacles \( T_1, \ldots, T_l \in T^+(V) \) with

\[
Z \deg(T_1) + \cdots + Z \deg(T_l) = Z \deg(T_1) \oplus \cdots \oplus Z \deg(T_l) \cong \mathbb{Z}^l.
\]

If there are no such tentacles, then we simply set \( c(V) = 0 \). Let \( K \subseteq V \) be a compact set with non-empty interior. Then \( T_K(V) \subseteq T(V) \) denotes the set of all elements \( T \in T(V) \) that have the same basis \( K \). In the same manner we define \( T^+_K(V) \subseteq T^+(V) \).

**Remark 1.2.2.** Let \( V \subseteq \mathbb{R}^n \) and consider the image of \( T^+(V) \) under the degree map \( \deg : T^+(V) \to \mathbb{Z}^n \). Then \( c(V) \) is bounded by the rank of the \( \mathbb{Z} \)-module that is generated by the image of \( T^+(V) \) under the degree map.

**Remark 1.2.3.** Let \( S \) be a semialgebraic set. With respect to \( c(S) \) we have the following observations:

- Let \( S \subseteq \mathbb{R}^n \) be a semialgebraic set. According to [8, Theorem 2.4.5, p. 35] the set \( S \) has a finite number \( S_1, \ldots, S_r \) of connected components that are also semialgebraic sets. Then

\[
c(S) \leq \sum_{i=1}^{r} c(S_i).
\]
• The inequality $c(S) \leq \dim(S)$ holds and is sometimes even strict: For example, $0 = c(B_1(0)) < \dim(B_1(0))$, where $B_1(0)$ is the closed $n$-dimensional unit-ball.

• For a semialgebraic set $S \subseteq \mathbb{R}^n$ we have $c(S) = n$ if and only if the $\mathbb{Q}$-vector space spanned by $\deg(T^+(S))$ is the whole space $\mathbb{Q}^n$: We will address only the less trivial direction. Suppose that $c(S) = n$. Then we get a exact sequence

$$0 \to \mathbb{Z}\deg(T_1) \oplus \cdots \oplus \mathbb{Z}\deg(T_n) \to \mathbb{Z}^{c(S)} \to 0.$$ 

Since $\mathbb{Q}$ is a localization of $\mathbb{Z}$ and therefore flat, we get the exact sequence

$$0 \to (\mathbb{Z}\deg(T_1) \oplus \cdots \oplus \mathbb{Z}\deg(T_n)) \otimes \mathbb{Q} \to (\mathbb{Z}^{c(S)}) \otimes \mathbb{Q} \to 0$$

resp.

$$0 \to \mathbb{Q}\deg(T_1) \oplus \cdots \oplus \mathbb{Q}\deg(T_n) \to \mathbb{Q}^{c(S)} \to 0.$$

Let us consider some rectangles $R_1, R_2, R_3, R_4, R_5 \subseteq \mathbb{R}^2$ as illustrated in the picture below.

Here $R_3, R_5, R_4$ and $R_2$ are infinite rectangles. Each rectangle is a semialgebraic set and therefore the union $S$ of these rectangles is a semialgebraic set. In each of the infinite rectangles we can place a tentacle of positive degree. For example, in $R_2$ we can place one of degree $(1,0)$, in $R_3$ of degree $(0,1)$, in $R_4$ of degree $(1,0)$ and in $R_5$ of degree $(0,1)$. As the shape of $S$ suggest, there are just two tentacles of these four tentacles that are fundamentally different. Obviously, $c(S) = 2$ because $\mathbb{Z}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \mathbb{Z}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cong \mathbb{Z}^2$.

However, we have $c(\mathbb{R}^2) = c(S)$, although $S$ and $\mathbb{R}^2$ are quite different. While $c(\mathbb{R}^2)$ and $c(S)$ do not point out this difference, the sets $T^+(S) \neq T^+(\mathbb{R}^2)$ obviously do.
Remark 1.2.4. Let us start where Remark 1.2.3 ended. Consider for example the semialgebraic set $S \subseteq \mathbb{R}^3$ that is depicted below:

![Diagram of a semialgebraic set]

While $R_1 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1, x_3 > 0, x_2 < 0\}$ and therefore $c(R_1) = 3$, the left upper infinite rectangle satisfies $c(R_2) = 2$. This can be detected by $T^+(R_1 \cup R_2)$: Fix a basis $K$ and consider $T^+_K(R_1 \cup R_2)$. For each basis $K$ we take the biggest number $l_K$ such that

$$\mathbb{Z} \deg(T_1) + \cdots + \mathbb{Z} \deg(T_{l_K}) = \mathbb{Z} \deg(T_1) \oplus \cdots \oplus \mathbb{Z} \deg(T_{l_K}) \cong \mathbb{Z}^{l_K}$$

for $T_1, \ldots, T_{l_K} \in \hat{T}_K(R_1 \cup R_2)$. In our situation the mapping $K \mapsto l_K$ is not constant. The difference with respect to $T^+(R_1 \cup R_2)$ and $c(R_1 \cup R_2)$ is that we are doing essentially the same thing but this time for a fixed basis $K$. The sets $T_K(S)$ for fixed basis $K$ together with the degree map can detect lower dimensional connected components of $S$.

One can refine the value $c(S)$ in various ways. For example, we could restrict ourselves just onto the asymptotic of a tentacle to define $c(S)$. Under this restriction the map $K \mapsto l_K$ is able to detect equal dimensional connected components of $S$.

Definition 1.2.5. Let $V \subseteq \mathbb{R}^n$. We say that $V$ is almost bounded if $c(V) = 0$.

Remark 1.2.6. Let $S \subseteq \mathbb{R}^n$ be a closed semialgebraic set such that $\overline{\text{int}(S)} = S$. Even under this new condition it is not clear if the following implication:

The set $S$ is almost bounded $\Rightarrow$ the set $S$ is bounded

holds. By [8] Proposition 8.1.13, p. 167] for every point $x \in S$ there is a NASH function $\varphi : (-1, 1) \rightarrow \mathbb{R}^n$ such that $\varphi(0) = x$ and $\varphi((0,1)) \subseteq S$. According to [8] Corollary 8.1.5, p. 165] Nash functions are locally analytic and therefore we cannot assume that a fiber of any tentacle can globally approximate the image $\varphi((0,1))$ within an acceptable margin of error. Thus, it could happen that $S$ is unbounded and satisfies $T^+(S) = \emptyset$. 

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Definition 1.2.7. Let $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n]$, $j \in \mathbb{N}$ and consider
\[
A_{r,j}(f_1, \ldots, f_r) = \begin{pmatrix}
  f_1(x_1, \ldots, x_n) & \cdots & f_1(x_{n-1}, \ldots, x_{nj}) \\
  \vdots & \ddots & \vdots \\
  f_r(x_1, \ldots, x_n) & \cdots & f_r(x_{n-1}, \ldots, x_{nj})
\end{pmatrix} \in \mathbb{R}[x_1, \ldots, x_{nj}]^{r \times j}.
\]

Let $\{e_1, \ldots, e_r\}$ be the standard basis of $\mathbb{R}^r$. The semialgebraic set $S(f_1, \ldots, f_r)$ is called \((r,j)-\text{ROW-COMPACT}\) if
\[
S_{\text{Row},r,j}(f_1, \ldots, f_r) := S \left( A_{r,j}(f_1, \ldots, f_r) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, e_1 \right), \ldots, \left( A_{r,j}(f_1, \ldots, f_r) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, e_r \right)
\]
is a compact semialgebraic set.

Remark 1.2.8. It is not true that the compactness of $S := S(f_1, \ldots, f_r)$ implies the compactness of $S_{\text{Row},r,r}(f_1, \ldots, f_r)$. In fact, $S_{\text{Row},r,r}(f_1, \ldots, f_r)$ does not even need to be almost bounded. For example, consider the two polynomials $f_1, f_2$ in $\mathbb{R}[x]$ with $f_1 = x$ and $f_2 = -x^3 + 100x^2$. Then $S(f_1, f_2)$ is compact. Let $\epsilon = \frac{1}{100}$. For $t \geq 1$ we get that
\[
 f_1(\epsilon + t) + f_1(-\epsilon - t) \geq 0.
\]
At the same time we have
\[
f_2(\epsilon + t) + f_2(-\epsilon - t) > 0.
\]
Thus, $S_{\text{Row},2,2}(f_1, f_2)$ is not compact and obviously $T^+(S_{\text{Row},2,2}(f_1, f_2))$ is not empty.

More generally, let $f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n]$ be polynomials such that
\begin{itemize}
  \item The semialgebraic set $S(f_1, \ldots, f_r)$ is compact.
  \item There are tentacles $T_i \in T^+(S(-f_i))$ and fibers $F_i = \text{Fibre}_{x_i}(T_i)$ for $i = 1, \ldots, r$ with unbounded smooth paths $\gamma_i : [0, \infty) \to F_i$ such that for each $i = 1, \ldots, r$ we have
    \[
    f_i(\gamma_1(t)) + \cdots + f_i(\gamma_r(t)) \to \infty
    \]
    and
    \[
    |f_i(\gamma_1(t))|, \ldots, |f_i(\gamma_r(t))| \to \infty
    \]
    for $t \to \infty$.
\end{itemize}

Obviously, the polynomials
\[
\left( A_{r,r}(f_1, \ldots, f_r) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, e_i \right)
\]
are not compact.
give rise to a semialgebraic set \( \text{SRow}_{r,r}(f_1, \ldots, f_r) \) that is not bounded. Furthermore, one can achieve \( T^+(\text{SRow}_{r,r}(f_1, \ldots, f_r)) \neq \emptyset \).

Motivated by the conditions above, we consider polynomials \( f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n] \) such that \( S(f_1, \ldots, f_r) \) and \( S = S(-f_1, \ldots, -f_r) \) are compact.

If we can find unbounded fibers \( F_i \subseteq S(-f_i) \) for \( i = 1, \ldots, r \) with \( F_j \not\subseteq S(-f_i) \) for \( j \neq i \), then \( S(f_1, \ldots, f_r) \) is a candidate for not being row-compact.

**Example 1.2.9.** The following semialgebraic sets are row-compact:

- The semialgebraic set \( S(1 - x_1^2 - x_2^2) \subseteq \mathbb{R}^2 \) is \((1, j)\)-row-compact for all \( j \in \mathbb{N} \).
- The semialgebraic set \( S(1 - x_1^2, 1 - x_2^2) \subseteq \mathbb{R}^2 \) is \((2, j)\)-row-compact for all \( j \in \mathbb{N} \).
- Let \( f_1, \ldots, f_r \in \mathbb{R}[x_1, \ldots, x_n] \) such that \( S(f_1), \ldots, S(f_r) \subseteq \mathbb{R}^n \) are compact. Then \( S(f_1, \ldots, f_r) \subseteq \mathbb{R}^n \) is \((r, j)\)-row-compact for all \( j \in \mathbb{N} \). Indeed, the sum
  \[
  \begin{pmatrix}
  A_{r,j}(f_1, \ldots, f_r) & 1 & \vdots & e_i
  \end{pmatrix}
  \]
  becomes negative for \( i = 1, \ldots, r \) if one variable becomes arbitrary large.
2 Free stability

One of the central and fundamental objects on which everything builds up is the POLYNOMIAL RING $\mathbb{C}(x_1, \ldots, x_n)$ of $n$ NON-COMMUTING indeterminate variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{C}$. This ring comes together with an involution $^\dagger$ which is determined by mapping a monomial $\alpha x_{i_1}^{a_{i_1}} \cdots x_{i_r}^{a_{i_r}}$ to $\alpha x_{i_r}^{a_{i_r}} \cdots x_{i_1}^{a_{i_1}}$ for $i_1, \ldots, i_r, a_{i_1}, \ldots, a_{i_r} \in \mathbb{N}$. In the commutative case it is quite clear 'where' to evaluate a polynomial: Usually one evaluates a polynomial on a algebraically closed field or a real closed field. Doing so with respect to non-commutative polynomials would not emphasize the fact that the variables do not commute anymore. Hence, it is convenient to evaluate a non-commutative polynomial by inserting objects out of the ring $M_k(\mathbb{C})$, the ring of $k \times k$ matrices with entries in $\mathbb{C}$. However, one can also insert more complicated objects like bounded operators on a Hilbert space $\mathcal{H}$ for example. A polynomial $f \in \mathbb{C}(x_1, \ldots, x_n)$ is called HERMITIAN if $f$ is invariant under $^\dagger$, i.e $f^\dagger = f$. The set of all such polynomials is denoted with $\mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$. Hermitian polynomials will usually be evaluated on spaces like $\text{Her}_k$ resp. $\text{Her}_k^+$, the set of all hermitian $k \times k$ matrices of $M_k(\mathbb{C})$ resp. the set of all positive semi-definite hermitian $k \times k$ matrices of $M_k(\mathbb{C})$.

2.1 Basic considerations

**Definition 2.1.1.** An algebra $A$ over some field $K$ is called LOCALLY FINITE GRADED if the following conditions are satisfied:

a The algebra $A$ has a grading $A = \bigoplus_{n=0}^{\infty} A_n$.

b The $K$-vector spaces $A_n$ are finite dimensional.

c The equality $A_0 = K$ holds.

The degree of an element $f \in A$ will be denoted with $\text{gr}(f)$.

**Example 2.1.2.** In the following we give two examples of the most important graded $\mathbb{C}$-algebras that we are using in this work.

The $\mathbb{C}$-algebra $\mathbb{C}[x_1, \ldots, x_n]$: For each $z \in \mathbb{Z}^n$ we get a grading of $\mathbb{C}[x_1, \ldots, x_n]$ by

$$\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d \in \mathbb{Z}} \mathbb{C}[x_1, \ldots, x_n]_{z,d}.$$
where \( \mathbb{C}[x_1, \ldots, x_n]_{z,d} = \{ \sum a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} : \sum \alpha_i z_i = d \} \). In fact, each \( z \in \mathbb{Z}^n \) gives rise to grading of \( \mathbb{C}[x_1, \ldots, x_n] \). If \( z = (1, \ldots, 1) \), then the induced grading is the standard grading of \( \mathbb{C}[x_1, \ldots, x_n] \). I.e. we have

\[
\mathbb{C}[x_1, \ldots, x_n] = \bigoplus_{d=0}^{\infty} \mathbb{C}[x_1, \ldots, x_n]_{d, d},
\]

where \( \mathbb{C}[x_1, \ldots, x_n]_{d, d} = \mathbb{C}[x_1, \ldots, x_n]_{(1, \ldots, 1), d} \). Under the standard grading \( \mathbb{C}[x_1, \ldots, x_n] \) becomes a locally finite graded \( \mathbb{C} \)-algebra.

The \( \mathbb{C} \)-algebra \( \mathbb{C}(x_1, \ldots, x_n) \): Let \( S_{d,n} \) be the set of all functions \( \{1, \ldots, d\} \rightarrow \{1, \ldots, n\} \). As in the commutative case we define the grading that is induced by some \( z \in \mathbb{Z}^n \) as

\[
\mathbb{C}(x_1, \ldots, x_n) = \bigoplus_{d \in \mathbb{Z}} \mathbb{C}(x_1, \ldots, x_n)_{z, d},
\]

with

\[
\mathbb{C}(x_1, \ldots, x_n)_{z, d} = \left\{ \sum_{j, \pi \in S_{d,n}} c_{\pi} x_{\pi(1)} \cdots x_{\pi(j)} : \sum_{i=1}^{j} z_{\pi(i)} = d \right\}.
\]

If \( z = (1, \ldots, 1) \), then we write

\[
\mathbb{C}(x_1, \ldots, x_n) = \bigoplus_{d=0}^{\infty} \mathbb{C}(x_1, \ldots, x_n)_{d, d},
\]

with \( \mathbb{C}(x_1, \ldots, x_n)_{d, d} = \mathbb{C}(x_1, \ldots, x_n)_{(1, \ldots, 1), d} \) and call it the standard grading of \( \mathbb{C}(x_1, \ldots, x_n) \). Under this grading \( \mathbb{C}(x_1, \ldots, x_n) \) becomes a locally finite graded \( \mathbb{C} \)-algebra.

**Definition 2.1.3.** The gradings of \( \mathbb{C}[x_1, \ldots, x_n] \) and \( \mathbb{C}(x_1, \ldots, x_n) \) that are induced by \( z \in \mathbb{Z} \) (see Example 2.1.2) are called \( z \)-gradings. If \( z = (1, \ldots, 1) \) the corresponding grading will be called the standard grading. For each element \( f \in \mathbb{C}[x_1, \ldots, x_n] \) the leading term with respect to the \( z \)-grading will be denoted with \( \text{Le}_z(f) \). If \( z = (1, \ldots, 1) \) we will just write \( \text{Le}(f) \).

In the same manner the leading term with respect to the \( z \)-grading in \( \mathbb{C}(x_1, \ldots, x_n) \) is defined: For each \( f \in \mathbb{C}(x_1, \ldots, x_n) \) the leading term is denoted with \( \text{Le}_z(f) \). If \( z = (1, \ldots, 1) \), then we erase the \( z \) in the notation.

The degree of a polynomial \( f \) with respect to some \( z \)-grading will be denoted with \( \text{gr}_z(f) \) in both cases, the commutative and the non-commutative case. In case of \( t = (1, \ldots, 1) \) we will simply write \( \text{deg} \) instead of \( \text{gr}_{(1, \ldots, 1)} \). The degree with respect to a \( z \)-grading will be called the \( z \)-degree. If \( z = (1, \ldots, 1) \) we will refer to the \( z \)-degree as the length.

**Remark 2.1.4.** The notations for the leading terms of \( \mathbb{C}[x_1, \ldots, x_n] \) and \( \mathbb{C}(x_1, \ldots, x_n) \) are the same. However, it will be always clear from the context which one is meant.
Remark 2.1.5. In the non-commutative situation the leading term $\text{Le}_z(f)$ of a polynomial $f \in \mathbb{C}(x_1, \ldots, x_n)$ has the following useful property: If $f$ is hermitian, then $\text{Le}_z(f)$ is also hermitian no matter which $z \in \mathbb{Z}^n$ we choose.

Definition 2.1.6. A free quadratic module generated by $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$ is given by

$$\text{QM}(f_1, \ldots, f_r) = \left\{ \sum_{i,j} q_{ij}^* f_i q_{ij} + s : q_{ij} \in \mathbb{C}(x_1, \ldots, x_n), s \in \Sigma \mathbb{C}(x_1, \ldots, x_n) \right\},$$

where $\Sigma \mathbb{C}(x_1, \ldots, x_n)^2 = \{ \sum_i q_i^* q_i : q_i \in \mathbb{C}(x_1, \ldots, x_n) \}$. In the same manner we define a quadratic module in the commutative sense: A quadratic module generated by polynomials $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]_{\text{her}}$ is given by

$$\text{QM}(f_1, \ldots, f_r) = \left\{ \sum_{i,j} s_{ij} \overline{s}_{ij} f_i + s : s_{ij} \in \mathbb{C}[x_1, \ldots, x_n], s \in \Sigma \mathbb{C}[x_1, \ldots, x_n]^2 \right\},$$

where $\Sigma \mathbb{C}[x_1, \ldots, x_n]^2 = \{ \sum_i s_i \overline{s}_i : s_i \in \mathbb{C}[x_1, \ldots, x_n] \}$.

Definition 2.1.7. We say a finitely generated free quadratic $M$ is totally stable with respect to an arbitrary grading, if $\deg(f) \leq \deg(f + g)$ for all $f, g \in M$. Total stability in the commutative case is defined entirely analogous.

Let $W$ be a finite dimensional $\ast$-subspace of $\mathbb{C}(x_1, \ldots, x_n)$. Let $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$. Then we write $\text{QM}_W(f_1, \ldots, f_r)$ for the set $\left\{ \sum_i \sum_j q_{ij}^* p_i q_{ij} + s : q_{ij} \in W, s \in \Sigma W^2 \right\}$. If $M = \text{QM}(f_1, \ldots, f_r)$ a quadratic module, then we set $M_W = \text{QM}_W(f_1, \ldots, f_r)$. The definition in the commutative case is entirely analogous.

Definition 2.1.8. A quadratic module $M = \text{QM}(f_1, \ldots, f_r)$ is said to be stable if for every finite dimensional $\ast$-subspace $U$ of $\mathbb{C}(x_1, \ldots, x_n)$ there is another finite dimensional $\ast$-subspace $W$ such that $M \cap U \subseteq M_W$.

Remark 2.1.9. Let $f, g \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$ and consider $\text{Le}(f)$ and $\text{Le}(g)$, the leading terms of $f$ and $g$ with respect to the standard grading. If any monomial in $\text{Le}(f)$ does not appear as a monomial in $\text{Le}(g)$ under any permutation, then it is obvious that $\text{gr}(f + g) \geq \text{gr}(f)$. This observation suggests that there are far more totally stable quadratic modules than in the commutative case.

As $\mathbb{C}[x_1, \ldots, x_n]$ can be identified with $\mathbb{C}[x] \otimes \cdots \otimes \mathbb{C}[x]$ and the ring $\mathbb{C}(x_1, \ldots, x_n)$ can be identified, in the language of [15], as the free product $\mathbb{C}[x] \ast \cdots \ast \mathbb{C}[x]$ over $\mathbb{C}$. By the universal property of the free product over $\mathbb{C}$, there is the canonical homomorphism

$$\psi : \mathbb{C}(x_1, \ldots, x_n) \to \mathbb{C}[x_1, \ldots, x_n]$$
that is given by \( x_i \mapsto x_i \) for \( i = 1, \ldots, n \). Conversely, there is some sort of canonical lift of any polynomial in \( \mathbb{C}[x_1, \ldots, x_n]_{\text{her}} \) into \( \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}} \). Motivated by the Weyl-Calculus in [29 Theorem 2.4, p. 19] we identify a polynomial

\[
\sum_{\alpha} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}[x_1, \ldots, x_n]_{\text{her}}
\]

by sending it to

\[
\sum_{\alpha} \left( \frac{1}{2} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} + \frac{1}{2} a_\alpha x_1^{\alpha_n} \cdots x_1^{\alpha_1} \right) \in \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}}.
\]

The kernel of \( \psi \) will be denoted with \( \tilde{3} \). Every \( f \in \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}} \) can be decomposed as \( f = g + r \) where \( r \in \tilde{3} \) and \( g \) satisfies \( \psi(g) = \psi(f) \). If \( g \), in addition, is a canonical lift from a polynomial in \( \mathbb{C}[x_1, \ldots, x_n]_{\text{her}} \), then the decomposition is called a standard decomposition.

If we have a quadratic module \( \mathcal{M}_{\text{nc}} = \text{QM}(f_1, \ldots, f_r) \) in \( \mathbb{C}\langle x_1, \ldots, x_n \rangle \), then we get a corresponding quadratic module \( \mathcal{M} = \text{QM}(\psi(f_1), \ldots, \psi(f_r)) \) in \( \mathbb{C}[x_1, \ldots, x_n] \). In future, we will not mention the mapping \( \psi \) and just write \( \text{QM}(f_1, \ldots, f_r) \) instead of \( \text{QM}(\psi(f_1), \ldots, \psi(f_r)) \).

Conversely, if we have a quadratic module \( \mathcal{M} = \text{QM}(f_1, \ldots, f_r) \) in \( \mathbb{C}[x_1, \ldots, x_n] \), then \( \mathcal{M}_{\text{nc}} = \text{QM}(f_1, \ldots, f_r) \) denotes the quadratic module in \( \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}} \) that we get by canonically identifying the polynomials \( f_1, \ldots, f_r \) in \( \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}} \).

**Lemma 2.1.10.** Let \( \mathcal{M} = \text{QM}(f_1, \ldots, f_r) \) be a quadratic module generated by the polynomials \( f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]_{\text{her}} \) and let \( \mathcal{M}_{\text{nc}} = \text{QM}(f_1', \ldots, f_r') \) be generated by the polynomials \( f_1', \ldots, f_r' \in \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}} \). Then the following statements hold:

a The quadratic module \( \mathcal{M} \) is totally stable with respect to some \( z \)-grading if and only if \( \text{QM}(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)) \) is totally stable with respect to the \( z \)-grading.

b The quadratic module \( \mathcal{M}_{\text{nc}} \) is totally stable with respect to some \( z \)-grading if and only if \( \text{QM}(\text{Le}_z(f_1'), \ldots, \text{Le}_z(f_r')) \) is totally stable with respect to the \( z \)-grading.

**Proof:** Obvious. \( \square \)

### 2.2 Non-commutative total stability with respect to commutative total stability

In Remark 2.1.9 we figured out that there must be far more totally stable non-commutative quadratic module than commutative totally stable quadratic modules. Suppose we have given \( \mathcal{M} = \text{QM}(f_1, \ldots, f_r) \). What can we say about \( \mathcal{M}_{\text{nc}} = \text{QM}(f_1, \ldots, f_r) \) and vice versa? If \( \mathcal{M}_{\text{nc}} \) is totally stable with respect to a \( z \)-grading, then it is convenient that we cannot tell anything about the total stability of the commutative counterpart. That
is because $\psi$ kills out all commutators and destroys total stability in most cases. Conversely, the total stability of $M$ with respect to a $z$-grading does not need to imply total stability for $M^{nc}$. In the following we will investigate these kinds of relationships.

**Definition 2.2.1.** Let $f \in \mathbb{C}(x_1, \ldots, x_n)$. For each monomial $m$ that appears in $f$, we define $\text{Term}(m)$ to be the corresponding term. Furthermore, we define $\text{Monomial}(f)$ to be the set of all monomials $m$ that appear in $f$ such that $\text{Term}(m) \neq 0$.

The condition $\text{Term}(m) \neq 0$ in Definition 2.2.1 is needed to assure that $\text{Monomial}(f)$ is well-defined. Without the second condition we would have $\text{Monomial}(x_1) \neq \text{Monomial}(x_1 + x_2 - x_2)$, while $x_1 = x_1 + x_2 - x_2$.

**Definition 2.2.2.** Let $f \in \mathbb{C}(x_1, \ldots, x_n)$. We say that $f$ divides another polynomial $g \in \mathbb{C}(x_1, \ldots, x_n)$ if there exist $h_1, h_2 \in \mathbb{C}(x_1, \ldots, x_n)\setminus\{0\}$ such that $h_1 f h_2 = g$. We say that $f$ divides $g$ by a square if $h_1 = h_2$.

**Definition 2.2.3.** Let $m, m' \in \mathbb{C}(x_1, \ldots, x_n)$ be two monomials. We say that $m'$ is a submonomial of $m$ if and only if $m'$ divides $m$ by a square.

In regard to stability, the following statement will give a sufficient condition for having total stability.

**Proposition 2.2.4.** Let $M^{nc} = QM(f_1, \ldots, f_r)$ be a quadratic module generated by the polynomials $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)\text{her}$ and endowed with a $z$-grading. Suppose that the following conditions are satisfied:

a) For $k \neq i$ and every $m \in \text{Monomial}(\text{Le}_z(f_i))$, $m' \in \text{Monomial}(\text{Le}_z(f_k))$ we have that neither $m$ does divide $m'$ by square nor that $m'$ divides $m$ by a square.

b) We have $\text{Le}_z \left( \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \right) \not\in -\Sigma(x_1, \ldots, x_n)^2$ whenever

$$\sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \neq 0$$

for $q_{ij} \in \mathbb{C}(x_1, \ldots, x_n)$. Then $M^{nc}$ is totally stable with respect to the $z$-grading.

**Proof:** The proof consists of two steps:

**Step I:** Consider the sum

$$f = \sum_{i,j} q_{ij}^* f_i q_{ij}$$

for $q_{ij} \in \mathbb{C}(x_1, \ldots, x_n)$. Using Lemma 2.1.10 we can assume that $f_i = \text{Le}_z(f_i)$ for all $i = 1, \ldots, r$, $q_{ij} = \text{Le}_z(q_{ij})$ for all $i, j$ and that $gr_z(q_{ij}^* f_i q_{ij})$ is the same number for all $i, j$.  

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Take a monomial

\[ a^*_i m a_{ij} \in \bigcup_{ij} \text{Monomial} \left( q^*_i f_{ij} q_{ij} \right) \]

of maximal length with \( a^*_i \in \text{Monomial}(q_{ij}) \) and \( m \in \text{Monomial}(f_i) \). Let \( a^*_i m' b^'_{ij} \) be another monomial with \( a^*_i m' b^'_{ij} \in \text{Monomial}(q_{ij}) \) such that \( m' \in \text{Monomial}(f_i) \) is of strictly higher length than \( m \). We will show that the term of \( a^*_i m' b^'_{ij} \) cannot terminate the term of \( a^*_i m a_{ij} \):

In order for \( a^*_i m' b^'_{ij} \) to terminate \( a^*_i m a_{ij} \), there must be submonomials \( e_1 \) of \( a^*_i \) and \( e_2 \) of \( a_{ij} \) such that \( m' = e_1 m e_2 \). Since \( m \) does not divide \( m' \) by a square the lengths of \( e_1 \) and \( e_2 \) cannot be the same.

Therefore, the lengths of \( a^*_i \) and \( b^'_{ij} \) cannot be the same. This implies that either

\[ \text{gr}(a^*_i, m' a^'_{ij}) > \text{gr}(a^*_i m a_{ij}) \]

or

\[ \text{gr}(b^*_i m' b^'_{ij}) > \text{gr}(a^*_i m a_{ij}) \]

contradicting the choice of \( a^*_i m a_{ij} \).

Suppose that the length of \( m' \) equals the length of \( m \). By maximality the lengths of \( a^*_i, a^*_i, b^*_{ij}, b^*_{ij} \) must all coincide. In order to terminate each other, the monomials \( m \) and \( m' \) must agree, implying that \( m \) divides \( m' \) by a square. But that would contradict our assumptions.

Therefore, we can assume that the lengths of the monomials \( m \) and \( m' \) are not equal. Let the length of \( m \) be strictly higher than the length of \( m' \). If \( m' \) is not a submonomial of \( m \), then we are already done: Maximality implies that the lengths of \( a^*_i, a^*_i, b^*_{ij}, b^*_{ij} \) must coincide. In order that the term belonging to \( a^*_i m a_{ij} \) is killed by the term belonging to \( a^*_i m' b^*_{ij} \), one of the monomials \( a^*_i \) or \( b^*_{ij} \) must contain a submonomial of \( m \), making its length strictly bigger than the length of \( a_{ij} \). But then \( a^*_i m' b^*_{ij} \) would exceed the length of \( a^*_i m a_{ij} \), which violates the maximality of \( a^*_i m a_{ij} \).

Suppose that \( m' \) is a submonomial of \( m \). Since \( m' \) cannot divide \( m \) by a square, there are \( c_1, c_2 \in \mathbb{C} \langle x_1, \ldots, x_n \rangle \) such that \( m = c_1 m' c_2 \) and \( c_1 \neq c_2 \). Suppose the associated terms would still terminate each other, i.e. \( a^*_i m a_{ij} = a^*_i m' c_2 a_{ij} \), where \( a^*_{ij} = (a^*_i c_1)^* \) and \( b^*_{ij} = c_2 a_{ij} \). Without loss of generality, we can assume that the length of \( c_1 \) is strictly greater than the length of \( c_2 \):
Let the lengths of $c_1$ and $c_2$ be the same. Then
\[ \text{gr}(b^{i,j}_m m' b^{i,j}) = \text{gr}(a^{i,j}_{m'} ma_{i,j}) = \text{gr}(a_{i,j}^{i,j} ma_{i,j}). \]

We want to know under which conditions can the monomial $a^{i,j}_{m'} ma_{i,j}$ be killed by some other monomial $a^{i,j}_{m''} m'' b^{i,j}$. If $m'$ is a submonomial of $m''$ or if neither $m'$ nor $m''$ are submonomials of each other, then the monomials cannot annihilate each other. So the only possibility left is that $m''$ is a submonomial of $m'$. In the case that $a^{i,j}_{m'} ma_{i,j} = -a^{i,j}_{m''} m'' b^{i,j}$ we simply repeat the same argument for $m''$ as we did for $m'$. After a finite amount of repetitions we will get a monomial, say $\tilde{m} \in \bigcup_i \text{Monomial}(f_i)$, such that $\tilde{m}$ does not have a non-trivial submonomial that also appears as a monomial of the $f_i$'s. Thus, for $\tilde{m}$ only the previous two possibilities, that we excluded, do apply. These arguments show why we can assume that $\text{gr}(c_1) \neq \text{gr}(c_2)$, because even if $\text{gr}(c_1) = \text{gr}(c_2)$ we can find another monomial where the corresponding term does not vanish.

Now $(a^{i,j}_{c_1} m' (a^{i,j}_{c_2} ma_{i,j})$ has a strictly higher length than $a^{i,j}_{c_1} ma_{i,j}$, resulting in a contradiction with the maximality of $a^{i,j}_{c_1} ma_{i,j}$.

**Step II:** Suppose that $f = \sum_{i,j} q_{ij} f_i q_{ij} + s$ for some $s \in \Sigma \mathbb{C}(x_1, \ldots, x_n)^2$. Since $\text{Le}_z \left( \sum_{i,j} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \right) \neq \Sigma \mathbb{C}(x_1, \ldots, x_n)^2$ and by what we have in step I before, we see that $\text{Le}_z \left( \sum_{i,j} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) + \text{Le}_z(s) \right) \neq 0$ and the assertion is proven. \[\square\]

**Remark 2.2.5.** Suppose that we have the same conditions as in Proposition 2.2.4. Let us assume that for every $m \in \bigcup_i \text{Monomial}(\text{Le}_z(f_i))$ we have that $m$ is not a square. Then condition (b) of Proposition 2.2.4 becomes obsolete. Consider the sum
\[ f = \sum_{i,j} q_{ij} f_i q_{ij} \]
from the proof of Proposition 2.2.4. We know that there is a $a \in \text{Monomial}(q_{ij})$ and a monomial $m' \in \bigcup_i \text{Monomial}(f_i)$ such that $a^* ma$ is contained in $\text{Monomial}(f)$ and is of maximal length. For a sum of squares $\sum s_i s_i$ the monomials that have the highest length are of the form $b^* b$, where $b \in \bigcup_i \text{Monomial}(s_i)$ has maximal length. Since $m$ is no square, there is no way that $-b^* b$ can annihilate $a^* ma$.

**Remark 2.2.6.** Suppose that condition (a) of Proposition 2.2.4 holds with respect to a $z$-grading. Let us have a closer look at condition (b) of Proposition 2.2.4. Let $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)$ be $z$-homogeneous polynomials such that for every $k \in \mathbb{N}$ there are points $(A_1, \ldots, A_n) \in \text{Her}^n_k$ and $(B_1, \ldots, B_n) \in \text{Her}^n_k$ with
\[ 0 \neq f_1(A_1, \ldots, A_n), \ldots, f_r(A_1, \ldots, A_n) \leq 0 \]
and
\[ 0 \neq f_1(B_1, \ldots, B_n), \ldots, f_r(B_1, \ldots, B_n) \geq 0. \]
By Theorem 1.1, p. 676 we have that

By condition (a) we get that

and

is the same number for all \( i, j \). We have

and

By condition (a) we get that

By [24, Theorem 1.1, p. 676] we have that

if and only if we can find a point \((A_1, \ldots, A_n) \in \text{Her}_k^n\) for a \( k \in \mathbb{N} \) such that

is not negative semi-definite.

Example 2.2.7. Let \( f_1 = -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2 - x_9^2, f_2 = -x_1^2 - x_2^2 - x_3^2 - x_4^2 - x_5^2 - x_6^2 - x_7^2 - x_8^2 - x_9^2 \). The leading terms \( \text{Le}(f_1) \) and \( \text{Le}(f_2) \) satisfy condition (a) of Proposition 2.2.4. All monomials in \( \text{Le}(f_1) \) and \( \text{Le}(f_2) \) are not squares and by Remark 2.2.5 condition (b) becomes obsolete. Thus, \( \text{QM}(f_1, f_2) \) is totally stable with respect to the standard grading. But if we consider the commutative version \( \text{QM}(f_1, f_2) = \text{QM}(-x_1^2) \), then it is certainly not true.

Remark 2.2.8. Proposition 2.2.4 can be viewed as a 'quadratic module' version of regular sequences of an ideal in the non-commutative case. To explain this connection let us assume that \( R \) is a commutative noetherian ring and \( I \) a proper ideal generated by elements \( x_1, \ldots, x_n \in R \). Furthermore we demand that for every quadratic module \( \text{QM}(f_1, \ldots, f_r) \) resp. \( \text{QM}(f_1, \ldots, f_r) \) that

resp.

\[
\text{Le}_z \left( \sum_{i,j} L_z(s_{ij}) \text{Le}_z(\pi_{ij}) \text{Le}_z(\psi(f_i)) \right) \notin \Sigma C[x_1, \ldots, x_n]^2
\]
whenever $\sum_{ij} \text{Le}_z(s_{ij})\text{Le}_z(\pi_{ij})\text{Le}_z(\psi(f_i)) \neq 0$ resp. $\sum_{ij} \text{Le}_z(q_{ij})\text{Le}_z(f_i)\text{Le}_z(q_{ij}) \neq 0$.

The cohomology of Koszul complexes allows us to detect the length of maximal $R$-sequences in $\mathcal{J}$. To be more precise, by [20, Theorem 17.4, p.428] we have the following situation: Let $\text{Kos}(x_1, \ldots, x_n)$ denote

$$\text{Kos}(x_1, \ldots, x_n) : 0 \to R \to N \to \wedge^2 N \xrightarrow{dx} \cdots \to \wedge^{n+1} N = 0,$$

for some $R$-module $N$ of rank $n$ and $d_x : \wedge^i N \to \wedge^{i+1} N$, $y \mapsto y\wedge x$ where $x = (x_1, \ldots, x_n)$. If $H^j(\text{Kos}(x_1, \ldots, x_n)) = 0$ for $j < r$ while $H^r(\text{Kos}(x_1, \ldots, x_n)) \neq 0$, then every maximal $R$-sequence in $\mathcal{J}$ has length $r$. If $R$ is a local ring then by [20, Theorem 17.6] we have the following refinement: If $H^{n-1}(\text{Kos}(x_1, \ldots, x_n)) = 0$, then $x_1, \ldots, x_n$ is a $R$-sequence. Using the fact that

$$H^n(\text{Kos}(x_1, \ldots, x_n)) \cong R/(x_1, \ldots, x_n)$$

and

$$H^j(\text{Kos}(x_1, \ldots, x_n)) \cong H_{n-j}(\text{Kos}(x_1, \ldots, x_n))$$

we get that the following statements are equivalent for a local ring $R$:

a. The sequence $x_1, \ldots, x_n$ is $R$-regular.

b. $H_0(\text{Kos}(x_1, \ldots, x_n)) \cong R/(x_1, \ldots, x_n)$ and $H_n(\text{Kos}(x_1, \ldots, x_n)) = 0$ for $n > 0$.

c. $H_1(\text{Kos}(x_1, \ldots, x_n)) = 0$.

Quadratic modules, however, are not lying in some local ring, but in the ring of multivariate polynomials over $\mathbb{C}$. If we want to relate the total stability of a quadratic module somehow to the statements $a, b$ and $c$, we must see what information about regular sequences are stored in these statements. In general the statements $a, b$ and $c$ cannot be used to detect the regular $R$-sequences. Only maximal regular $R$-sequences can be detected, since $\text{depth}(\mathcal{J}, R) = \text{depth}(\mathcal{J}, R_\mathfrak{p})$ for some prime ideal $\mathfrak{p}$ contained in $\text{supp}(R)$. In case of maximal sequences we can still make use of $a, b$ and $c$. Let us return to the study of a quadratic module $\text{QM}(f_1, \ldots, f_r)$, where $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous with respect to some $z$-grading. Even if any permutation of $f_1, \ldots, f_r$ is a regular $\mathbb{C}[x_1, \ldots, x_n]$-sequence, we cannot conclude that

$$\text{gr}_x \left( \sum_{ij} s_{ij} \tilde{s}_{ij} f_i \right) = \max \{\text{gr}_x (s_{ij} \tilde{s}_{ij} f_i) \}$$

since $s_{ij} \tilde{s}_{ij}$ might vanish under the projection mapping

$$\mathbb{C}[x_1, \ldots, x_n] \to \mathbb{C}[x_1, \ldots, x_n]/(f_1, \ldots, f_r).$$

So we see that even under these conditions we cannot assume that $\text{QM}(f_1, \ldots, f_r)$ is totally stable.
Interestingly, things change if we consider the non-commutative ring \( \mathbb{C}\langle x_1, \ldots, x_n \rangle \). The details are contained in [4] and we will just give a short summary. In order to demonstrate this connection let \( R \) be a locally finitely graded algebra over some field \( \mathbb{K} \). If \( R \) is commutative and \( x_1, \ldots, x_n \) a regular \( R \)-sequence then this is equivalent to having an isomorphism

\[
R \cong \mathbb{K}[x_1, \ldots, x_n] \otimes R/(x_1, \ldots, x_n)
\]
/of graded algebras. If \( R \) is not commutative, then it is convenient to call a sequence \( x_1, \ldots, x_n \) \( \text{regular} \) in \( R \), if there is an isomorphism

\[
R \cong \mathbb{K}\langle x_1, \ldots, x_n \rangle \ast R/I,
\]

where \( I \) is the two-sided ideal generated by \( x_1, \ldots, x_n \). We need to define a non-commutative analogon of the Koszul homology. Let \( x_1, \ldots, x_n \in R \) be elements with positive degree with respect to the grading. For the sequence \( x_1, \ldots, x_n \) we introduce new variables \( y_1, \ldots, y_n \) such that the degree of \( y_i \) is higher by one than the degree of \( x_i \). This gives rise to a graded algebra

\[
\mathcal{A} = R \ast \mathbb{K}\langle y_1, \ldots, y_n \rangle.
\]

Let \( d : \mathcal{A} \to \mathcal{A} \) be the derivation that is defined by the conditions \( d(y_i) = x_i \) for \( i = 1, \ldots, n \) and \( d^2 = d \circ h = 0 \), where \( h : \mathcal{A} \to R \) is the canonical homomorphism that is given by \( h(y_i) = 1 \) for \( i = 1, \ldots, r \). Since \( d^2 = 0 \), the map \( d \) is a boundary map. Now, \( d \) maps any element of \( R \) to 0 and therefore we need a new grading in order to be compatible with \( d \). On \( \mathcal{A} \) we introduce a new grading by setting the degree of all elements in \( R \) to zero and the degree of the new elements \( y_1, \ldots, y_n \) to 1, giving\(^2\)

\[
\mathcal{A} = R \ast \bigoplus_{i \geq 0} \mathbb{K}\langle y_1, \ldots, y_n \rangle_i =: \bigoplus_{i \geq 0} \mathcal{A}_i.
\]

Putting all the information together results in the complex

\[
\cdots \to \mathcal{A}_i \xrightarrow{d} \mathcal{A}_{i-1} \xrightarrow{h} \mathcal{A}_1 \to R \to 0.
\]

The above complex is denoted with \( \text{Kos}(x_1, \ldots, x_n, y_1, \ldots, y_n) \) to emphasize its role as the new Koszul-complex. By [4, Theorem 2.8, p. 128] we have the following equivalent statements:

\(^2\)Let \( g = y_0y_1 \cdots y_ly_0 \in \mathcal{A} \). The new grading of \( \mathcal{A} \) is defined in such a way that the degree of \( d(g) \) decreases by one compared to the degree of \( g \).
A. The sequence $x_1, \ldots, x_n$ is regular in $R$.

B. $H_0(\text{Kos}(x_1, \ldots, x_n, y_1, \ldots, y_n)) = R/\mathcal{I}$ and $H_n(\text{Kos}(x_1, \ldots, x_n, y_1, \ldots, y_n)) = 0$ for $n > 0$.

C. $H_1(\text{Kos}(x_1, \ldots, x_n, y_1, \ldots, y_n)) = 0$.

Note that conditions A-C perfectly resemble conditions a-c.

With the help of Proposition 2.2.4 and [4] Theorem 3.1, p. 134] we will describe how regular sequences are connected with the total stability of a quadratic module:

Under the assumption that we have $R = \mathbb{C}(x_1, \ldots, x_n)$, a sequence of monomials $m_1, \ldots, m_r \notin \mathbb{C}$ is regular if and only if it is COMBINATORIALLY FREE. Combinatorially free means that no $m_i$ is a submonomial of $m_j$ for $i \neq j$ and whenever $m_i = x_1 y_1$ and $m_j = x_2 y_2$ for some monomials $x_1, y_1, x_2, y_2 \in \mathbb{K}$ we have $x_1 \neq y_2$. It is easy to find monomials $m_1, \ldots, m_r \in R$ that satisfy Proposition 2.2.4 but will fail to be combinatorial free. Hence, the first homology of the corresponding complex will not vanish, while $\text{QM}(m_1, \ldots, m_r)$ will be totally stable with respect to the $z$-grading. Finally, let us consider the case where we are dealing with polynomials instead of just monomials. Invoking [4] Theorem 3.2, p. 135] we have the following implication: The sequence $f_1, \ldots, f_r \in R\setminus\{0\}$ is regular if the leading monomials $m_1, \ldots, m_r \in R$ of $f_1, \ldots, f_r$ with respect to the MONOMIAL ORDERING $\leq_{\text{mon}}$ are combinatorially free. This monomial ordering $\leq_{\text{mon}}$ on $\mathbb{C}(x_1, \ldots, x_n)$ is characterized by the following properties:

- If we have two monomials $m, m'$ with $\text{gr}(m) < \text{gr}(m')$, then $m <_{\text{mon}} m'$.
- If we have two monomials $m, m'$ with $\text{gr}(m) = \text{gr}(m')$, then $m \leq_{\text{mon}} m'$ if and only if $m$ is smaller than $m'$ with respect to the LEXICOGRAPHICAL ORDERING.

The work that has been done so far in Remark 2.2.8] will lead straight to Theorem 2.2.10]

The necessary details will be presented below:

**Lemma 2.2.9.** Let $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)_{\text{hom}} \setminus \mathbb{R}$ and let $m_1, \ldots, m_r$ be the leading monomials of $\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)$ with respect to $\leq_{\text{mon}}$. If the monomials $m_1, \ldots, m_r$ are combinatorially free, then

$$\text{gr}_z \left( \sum_{i,j} q^*_i \text{Le}_z(f_i)q_{ij} \right) = \max \left\{ \text{gr}_z \left( q^*_i \text{Le}_z(f_i)q_{ij} \right) \right\}$$

for any $q_{ij} \in \mathbb{C}(x_1, \ldots, x_n)$.

**Proof:** Let $m_i \in \text{Monomial}(\text{Le}_z(f_i))$ be the leading monomials with respect to some fixed monomial ordering for $i = 1, \ldots, r$. For every $i \neq j$ we have that neither $m_i$ is a submonomial of $m_j$ nor that $m_j$ is a submonomial of $m_i$. Let $q_{ij} \in \mathbb{C}(x_1, \ldots, x_n)$ and let $m^r_{ij} \in \text{Monomial}(\text{Le}_z(q_{ij})^*\text{Le}_z(f_i)\text{Le}_z(q_{ij}))$ be the leading monomials with respect to
\leq_{\text{mon}}. Each \( m'_{ij} \) can be written as \( m'_{ij} = d_{ij}^t m_i c_{ij} \), where \( c_{ij}, d_{ij} \in \text{Monomial}(q_{ij}) \). By definition of \( \leq_{\text{mon}} \) we have that \( \text{gr}(d_{ij}) = \text{gr}(c_{ij}) \). Let \( m_{ij} \) and \( m'_{ij} \) be two monomials of maximal length. Since \( m_i \) is not a submonomial of \( m' \), resp. \( m'_{ij} \) is no submonomial of \( m_i \), we get that the terms of \( m_{ij} \) and \( m'_{ij} \) cannot annihilate each other. Thus

\[
\text{gr}_z \left( \sum_{ij} \text{Le}_z(q_{ij})^t \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \right) = \max \{ \text{gr}_z \left( \text{Term}(m_{ij}) \right) \} = \max \{ \text{gr}_z \left( q_{ij}^t \text{Le}_z(f_i)q_{ij} \right) \}
\]

and we are done. \( \square \)

**Theorem 2.2.10.** Let \( f_1, \ldots, f_r \in \mathbb{C} \langle x_1, \ldots, x_n \rangle \) and let \( m_i \in \text{Monomial}(\text{Le}_z(f_i)) \) be the leading monomials with respect to \( \leq_{\text{mon}} \) for \( i = 1, \ldots, r \). Suppose that one of the equivalent statements hold:

a) The sequence \( m_1, \ldots, m_r \) is regular.

b) The monomials \( m_1, \ldots, m_r \) are combinatorially free.

c) We have \( H_i(\text{Kos}(m_1, \ldots, m_r, y_1, \ldots, y_r)) = 0 \).

Then

\[
\text{gr}_z \left( \sum_{ij} q_{ij}^t \text{Le}_z(f_i)q_{ij} \right) = \max \{ \text{gr}_z \left( q_{ij}^t \text{Le}_z(f_i)q_{ij} \right) \}
\]

for all \( q_{ij} \in \mathbb{C} \langle x_1, \ldots, x_n \rangle \). If in addition \( \text{Le}_z \left( \sum_{ij} q_{ij}^t \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \right) \not\in -\Sigma \mathbb{C} \langle x_1, \ldots, x_n \rangle^2 \) holds whenever \( \sum_{ij} q_{ij}^t \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \neq 0 \), then

\[
\text{QM}(f_1, \ldots, f_r)
\]

is totally stable with respect to the \( z \)-grading.

**Proof:** By what we have done in Remark 2.2.8 we already know that the statements (a)-(c) are equivalent. Together with Lemma 2.2.9 it is obvious that the statement of the proposition holds. \( \square \)

**Lemma 2.2.11.** Let \( M = \text{QM}(f_1, \ldots, f_r) \) be a quadratic module generated by the polynomials \( f_1, \ldots, f_r \in \mathbb{C} \langle x_1, \ldots, x_n \rangle \) and suppose that the following conditions are met:

a) Let \( \text{Le}_z(f_i) = g_i + r_i \) for \( g_i, r_i \in \mathbb{C} \langle x_1, \ldots, x_n \rangle \) and \( r_i \in \mathbb{N} \) be a standard decomposition for \( i = 1, \ldots, r \) such that \( \text{gr}(g_i) > \text{gr}(r_i) \).

b) We have \( \text{Le}_z \left( \sum_{ij} q_{ij}^t \text{Le}_z(s_{ij}) \text{Le}_z(\overline{s_{ij}}) \text{Le}_z(\psi(f_i)) \right) \not\in -\Sigma \mathbb{C} [x_1, \ldots, x_n]^2 \) whenever \( \sum_{ij} q_{ij}^t \text{Le}_z(s_{ij}) \text{Le}_z(\overline{s_{ij}}) \text{Le}_z(\psi(f_i)) \neq 0 \) for \( s_{ij} \in \mathbb{C}[x_1, \ldots, x_n] \).
c The semialgebraic sets

\[ S'(\text{Le}(\psi(f_1)), \ldots, \text{Le}(\psi(f_r))), S'(\text{Le}(\psi(f_1)), \ldots, \text{Le}(\psi(f_r))) \]

are not empty.

Then

\[ \text{Le}_z \left( \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \right) \notin \mathbb{D}\mathbb{C}(x_1, \ldots, x_n)^2 \]

whenever \( \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \neq 0 \) for \( q_{ij} \in \mathbb{C}(x_1, \ldots, x_n) \).

**Proof:** Suppose the assertion would be false, i.e. we can find \( q_{ij}, q_{ij}' \in \mathbb{C}(x_1, \ldots, x_n) \) such that \( \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \neq 0 \) and

\[ \text{Le}_z \left( \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) \right) + \sum_i q_i^* q_i' = 0. \]

Without loss of generality we can assume that the degrees \( \text{gr}_z(\text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij})) \) are all equal for \( i, j \). So we get the equation

\[ \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) + \sum_i q_i^* q_i' = 0. \tag{2.1} \]

Furthermore, we can assume that the polynomials \( q_{ij}, f_i \) and \( q_{ij}' \) are all \( z \)-homogeneous.

**Step I: Reduction:** Let \( q_{ij} = w_{ij} + d_{ij} \) be a standard decomposition with \( w_{ij} \in \mathbb{C}(x_1, \ldots, x_n) \) and \( d_{ij} \in \mathbb{Z} \) for all \( i, j \). In the same manner, we write \( q_{ij}' = w_{ij}' + d_{ij}' \) and \( f_i = g_i + r_i \) with \( d_{ij}', r_i \in \mathbb{Z} \). Plugging this into the equation above leads to

\[ \sum_{ij} w_{ij}' g_{ij} + R + \sum_i w_i^* w_i' + Q = 0, \]

where

\[ R = \sum_{ij} q_{ij}' f_i q_{ij} - \sum_{ij} w_{ij}' g_{ij} \]

and

\[ Q = \sum_i q_i^* q_i' - \sum_i w_i^* w_i'. \]

Applying \( \psi \) to the equation above leads to

\[ \psi \left( \sum_{ij} w_{ij}' g_{ij} \right) + \psi \left( \sum_i w_i^* w_i' \right) = 0, \]

since \( R, Q \in \mathbb{Z} \). Conditions (b) and (c) imply that all the polynomials \( w_i' \) and \( w_{ij} \) must vanish: If not all \( w_i' \) vanish, then \( \psi(\sum_i w_i^* w_i') \neq 0 \) which implies \( \psi \left( \sum_{ij} w_{ij}' g_{ij} \right) \in \mathbb{Z} \).
This observation leads us to the equation

$$R + Q = \sum_{ij} d_{ij}^g d_{ij} + \sum_{ij} d_{ij}^r d_{ij} + \sum_i d_i^* d_i = 0. \quad (2.2)$$

To summarize up, we reduced equation (2.1) to equation (2.2) and reduced

$$\sum_{ij} q_{ij}^i f_{ij} q_{ij}$$

to $R'$.

Step II: Generating a contradiction: The idea is to show that equation (2.1) and conditions (c) and (a) cannot hold at the same time. By step I we can use the reduced equation (2.2). By conditions (a) and (c) we have

$$\text{Le}(R') = \text{Le} \left( \sum_{ij} d_{ij}^g d_{ij} + \sum_{ij} d_{ij}^r d_{ij} \right) = \text{Le} \left( \sum_{ij} d_{ij}^g d_{ij} \right): \quad (2.3)$$

Let $m = \max\{\text{gr}(d_{ij}^g d_{ij}) : i, j\}$ and $I$ the set of all indices $(i', j')$ such that $\text{gr}(d_{i'j'}^g d_{i'j'}) = m$. If we can show that

$$\sum_{(i', j') \in I} \text{Le}(d_{i'j'}^*) \text{Le}(g_{i'}) \text{Le}(d_{i'j'}) \neq 0,$$

then equation (2.3) must hold by condition (a). Let $(A_1, \ldots, A_n) \in \text{Her}_k^k$ be given by

$$A_1 = \begin{pmatrix} y & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y \end{pmatrix}, \ldots, A_n = \begin{pmatrix} y & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & y \end{pmatrix},$$

where $y \in S'(\text{Le}(\psi(f_1)), \ldots, \text{Le}(\psi(f_r)))$. There is a neighborhood $U$ of $(A_1, \ldots, A_n)$ such that $g_i$ is positive on $U$ for all $i = 1, \ldots, r$. Then it is not difficult to see that $\sum_{(i', j') \in I} \text{Le}(d_{i'j'}^*) \text{Le}(g_{i'}) \text{Le}(d_{i'j'})$ vanishes on $U$ for all $k \in \mathbb{N}$ if and only if $d_{i'j'} = 0$ for all $(i', j') \in I$. This statement and the definition of $I$ give us the non-vanishing of $\sum_{(i', j') \in I} \text{Le}(d_{i'j'}^*) \text{Le}(g_{i'}) \text{Le}(d_{i'j'})$.

Combining equation (2.2) and equation (2.3) leads straight to

$$\text{Le} \left( \sum_{ij} d_{ij}^g d_{ij} \right) + \text{Le} \left( \sum_i d_i^* d_i \right) = 0. \quad (2.4)$$

Equation (2.4) is obviously incompatible with condition (c).

\[\square\]

\[3\] Otherwise, condition (c) shows that $\sum_{ij} \psi(w_{ij}) \psi(w_{ij}) \psi(g_i)$ is not the zero polynomial.
Remark 2.2.12. Suppose we replace condition (a) of Lemma 2.2.11 by the following one:

(a') Let $L_z(f_i) = g_i + r_i$ for $g_i \in \mathbb{C}(x_1, \ldots, x_n)$ and $r_i \in \mathfrak{J}$ be a standard decomposition for $i = 1, \ldots, r$. Suppose that we have $r_i = 0$ for all $i = 1, \ldots, r$.

Furthermore, we replace condition (c) by the following new one:

(c') The semialgebraic set $S(L_z(\psi(f_1)), \ldots, L_z(\psi(f_r)))$ is not empty.

In other words, if the $z$-degree of the '$3$-part' of $L_z(f_i)$ vanishes, then the proof of Lemma 2.2.11 is much easier. Indeed, under condition (a') equation 2.2 becomes

$$R + Q = \sum_{ij} d_{ij}g_i d_{ij} + \sum_i d_i r_i = 0.$$ 

It is immediately clear that this equation is not compatible with condition (c').

Proposition 2.2.13. Let $M_{nc} = QM(f_1, \ldots, f_r)$ be a quadratic module generated by the polynomials $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$ and endowed with some $z$-grading. Suppose that the following conditions are satisfied:

- For $k \neq i$ and any $m \in \text{Monomial}(L_z(f_i))$, $m' \in \text{Monomial}(L_z(f_k))$ we have that neither $m$ does divide $m'$ by square nor that $m'$ divides $m$ by a square.

- Let $L_z(f_i) = g_i + r_i$ for $g_i \in \mathbb{C}(x_1, \ldots, x_n)$ and $r_i \in \mathfrak{J}$ be a standard decomposition for $i = 1, \ldots, r$ such that $\text{gr}(g_i) > \text{gr}(r_i)$.

- We have $L_z\left(\sum_{ij} L_z(s_{ij}) L_z(x_{ij}) L_z(\psi(f_i))\right) \notin -\Sigma[\mathfrak{J}] [x_1, \ldots, x_n]_2$ whenever $\sum_{ij} L_z(s_{ij}) L_z(x_{ij}) L_z(\psi(f_i)) \neq 0$ for $s_{ij} \in \mathbb{C}[x_1, \ldots, x_n]$.

- The semialgebraic sets $S'(L_z(\psi(f_1)), \ldots, L_z(\psi(f_r))), S'(L_z(\psi(f_1)), \ldots, L_z(\psi(f_r)))$ are not empty.

Then $M_{nc}$ is totally stable with respect to the $z$-grading.

**Proof:** Combination of Proposition 2.2.4 and Lemma 2.2.11
2.3 Stability with respect to the Weyl Algebra

Given a non-commutative quadratic module $M^{nc}$ we saw in the form of Proposition 2.2.13 that the commutative version $M$ also stores some information about $M^{nc}$. The question is, if we take a two-sided ideal $\mathcal{I}$ and consider the ring $\mathbb{C}\langle x_1, \ldots, x_n \rangle / \mathcal{I}$, does the commutative module then store more information about the non-commutative? The trivial answer is yes, because we can take $\mathcal{I} = \mathbb{C}$. In the following we want to find a non-trivial ideal $\mathcal{I}$ such that the total stability of finitely generated modules in $\mathbb{C}\langle x_1, \ldots, x_n \rangle / \mathcal{I}$ behaves as close as possible with respect to the trivial case. To make things easier we will switch to the $\ast$-algebra $\mathbb{C}\langle x_1, \ldots, x_n, x_1^\ast, \ldots, x_n^\ast \rangle$. Let $\mathfrak{M}$ be the two-sided ideal generated by the elements

$$x_i x_j^\ast - x_j^\ast x_i - \Delta_{ij},$$

where

$$\Delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

for $i, j = 1 \ldots, n$. The quotient $\mathbb{C}\langle x_1, \ldots, x_n, x_1^\ast, \ldots, x_n^\ast \rangle / \mathfrak{M}$ is called the $n$-th Weyl algebra and is denoted with $\mathcal{W}_n$. The Weyl Algebra has a natural filtration and this filtration has $\mathbb{C}[x_1, \ldots, x_n, x_1^\ast, \ldots, x_n^\ast]$ as its associated graded algebra. Let $\Sigma W_n^2$ denote the set of all sums of squares of $\mathcal{W}_n$, i.e. the elements of $\Sigma W_n^2$ are of the form $\sum_i f_i^2 f_i$ for $f_i \in \mathcal{W}_n$. All sums of squares of $\mathcal{W}_n$ are contained in the set $\mathcal{W}_n, n$ of all hermitian elements of $\mathcal{W}_n$. Any $z$-grading of $\mathbb{C}\langle x_1, \ldots, x_{2n} \rangle$ can be transferred to $\mathbb{C}\langle x_1, \ldots, x_n, x_1^\ast, \ldots, x_n^\ast \rangle$ by the following procedure: It is pretty straight forward to show that any element $h \in \mathcal{W}_n$ can uniquely be written as

$$h = \sum_{\alpha} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_1^{\ast \alpha_n+1} \cdots x_n^{\ast \alpha_n}$$

for $a_\alpha \in \mathbb{C}$. It is then convenient to define the $z$-degree on $\mathcal{W}_n$ by

$$\overline{gr}_z(h) = gr_{z} \left( \sum_{\alpha} a_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n} x_1^{\ast \alpha_n+1} \cdots x_n^{\ast \alpha_n} \right).$$

Finally, $\mathcal{W}_n$ has one distinguished faithful $\ast$-representation $\pi_0$ acting on the Schwartz space $S(\mathbb{R}^n)$. In contrast to the commutative case we get the following Positivstellelsatz:

**Theorem 2.3.1.** Let $f \in \mathcal{W}_n, n$ of degree $2m$ and let $\text{Le}(f)$ be the leading term of $f$. Let $g$ be the polynomial in $\mathbb{C}[x_1, \ldots, x_n, x_1^\ast, \ldots, x_n^\ast]$ that is associated with $\text{Le}(f)$. Assume that there exists $\varepsilon > 0$ such that

$$\langle \pi_0(g - \varepsilon) \varphi, \varphi \rangle \geq 0$$

for all $\varphi \in S(\mathbb{R}^n)$. If $m$ is even, then there is an element $b$ which is a finite product of elements of the form $x_1^{\alpha_1} x_1 + \cdots + x_n^{\alpha_n} x_n + (a + k)$, $k \in \mathbb{Z}$ such that $b f b \in \Sigma W_n^2$. If $m$ is odd, we just have $\sum_{j=1}^{\frac{n}{2}} b x_j f x_j^\ast b \in \Sigma W_n^2$.
Proof: See [50, Theorem 1.1, p. 2].

Proposition 2.3.2. The generators of the ideal \( \mathfrak{m} \) form a regular sequence.

Proof: That follows from [4, Theorem 3.2, p. 135].

Proposition 2.3.3. Let \( f_1, \ldots, f_r \) be a regular sequence in \( \mathbb{C}(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*) \) that contains the generators of \( \mathfrak{m} \) as a subsequence. Let \( f_{i_1}, \ldots, f_{i_j} \) be the sequence that we get from expelling all generators of \( \mathfrak{m} \). Then \( f_{i_1}, \ldots, f_{i_j} \) is a regular sequence in \( \mathcal{W}_n \).

Proof: That is Proposition 2.3.2 and [4, Lemma 2.7, p. 126].

Remark 2.3.4. Note that one cannot expect the generators of \( \mathfrak{m} \), interpreted as elements of \( \mathbb{C}(x_1, \ldots, x_{2n}) \) via the \(*\)-isomorphism

\[
\mathbb{C}(x_1, \ldots, x_n, x_1^*, \ldots, x_n^*) \xrightarrow{\sim} \mathbb{C}(x_1, \ldots, x_{2n}), x_j \mapsto x_j + i x_{n+j}, x_j^* \mapsto x_j - i x_{n+j},
\]

to constitute a regular sequence again: Using the \(*\)-isomorphism we get

\[
x_k x_j - i x_k x_{n+j} + i x_{n+k} x_j + x_{n+k} x_{n+j} - x_j x_k - i x_{n+j} x_k - i x_{n+j} x_{n+k} - \Delta_{k,j}.
\]

By using the lexicographical ordering we see that [4, Theorem 3.2, p. 135] cannot be applied. To be more precise, for \( k < j \) we have \( x_k x_j \) as the leading monomial and for \( j < k \) we have also \( x_j x_k \) as the leading monomial, since both appear in every polynomial above.

Proposition 2.3.5. Let \( z \in \mathbb{N}^n \). The following statements for a Weyl algebra \( \mathcal{W}_n \) hold:

a For \( f, g \in \mathcal{W}_n \) we have \( \text{gr}_z(fg) = \text{gr}_z(f) + \text{gr}_z(g) \).

b For \( f, g \in \Sigma \mathcal{W}_n \) we have \( \text{gr}_z(f + g) = \max \{ \text{gr}_z(f), \text{gr}_z(g) \} \).

c The quadratic module \( \Sigma \mathcal{W}_n \) is totally stable with respect to any \( z \)-grading.

Proof: Statement (c) follows directly from (a) and (b). Therefore, we will just prove statements (a) and (b).

We need the following preliminary:

For \( h_1, h_2 \in \mathcal{W}_n \) we have \( \text{gr}_z(h_1 \cdot h_2) = \text{gr}_z(h_1) + \text{gr}_z(h_2) \): We will consider the case where \( h_1, h_2 \in \mathcal{W}_1 \) as the assertion follows inductively from this case. Let \( x^i x^j \) be a monomial that appears in \( \text{Le}_z(h_1) \) and let \( x^i x^j \) be one that appears in \( \text{Le}_z(h_2) \). Then

\[
x^i x^j \cdot x^k x^l = \sum_{r=0}^{j} r! \binom{j}{r} \binom{k}{r} x^{i+k-r} x^{j+l-r}.
\]

Now we are done because \( i z_1 + k z_1 + j z_2 + l z_2 = \text{gr}_z(h_1) + \text{gr}_z(h_2) \) and the above monomial appears in \( \text{Le}_z(h_1 \cdot h_2) \). Now we are done:

(a): That is obvious.

(b). Follows directly by sorting the monomials of \( \text{Le}_z(f + g) \).
Theorem 2.3.6. Let $\tilde{M}^{nc}$ be a finitely generated quadratic module in $\mathcal{W}_n$ and $\tilde{M}$ the corresponding one in the associated graded algebra $\mathbb{C}[x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}]$ and $z \in \mathbb{N}^n$. Then $\tilde{M}^{nc}$ is totally stable with respect to the $z$-grading if and only if $\tilde{M}$ is totally stable with respect to the corresponding $z$-grading of $\mathbb{C}[x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}]$.

Proof: Suppose that we have a $z$-filtration

$$((\mathcal{W}_n)_{z,d})_d$$

with the associated algebra

$$\bigoplus_{d \in \mathbb{N} \cup \{0\}} (\mathcal{W}_n)_{z,d}/(\mathcal{W}_n)_{z,d-1}.$$

For two homogeneous elements $f, g \in \mathcal{W}_n$ we have seen that $\mathcal{g}^{gr}_z(fg) = \mathcal{g}^{gr}_z(f) + \mathcal{g}^{gr}_z(g)$. The multiplication even behaves as in $\mathbb{C}[x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}]$. Hence, we have an isomorphism

$$\bigoplus_{d \in \mathbb{N} \cup \{0\}} (\mathcal{W}_n)_{z,d}/(\mathcal{W}_n)_{z,d-1} \cong \bigoplus_{d \in \mathbb{N} \cup \{0\}} \mathbb{C}[x_1, \ldots, x_n, \overline{x_1}, \ldots, \overline{x_n}]_{z,d}$$

of degree 0. Invoking Lemma 2.1.10 leads straight to the conclusion that $\tilde{M}^{nc}$ it totally stable with respect to the $z$-grading if and only if $\tilde{M}$ is totally stable with respect to the $z$-grading.

Remark 2.3.7. Let us summarize the implications of what we have done about the Weyl algebra:

- By using Remark 2.2.8 and Proposition 2.3.2 we see that the Weyl algebra $\mathcal{W}_n$ appears as the 0-th Koszul homology of $\mathbb{C}[x_1, \ldots, x_{2n}]$ with respect to the regular sequence that is given by the generators of the ideal $\mathcal{W}_n$.

- Theorem 2.3.6 can be viewed as a much stronger version of Proposition 2.2.13: While under certain conditions Proposition 2.2.13 allows us to replace a non-commutative with a commutative condition, Theorem 2.3.6 tells us that the total stability in $\mathcal{W}_n$ behaves as one expects in a commutative ring.

- Proposition 2.3.3 tells us that certain regular sequences remain regular in $\mathcal{W}_n$. Remark 2.3.4 tells us that under the $*$-isomorphism

$$\mathbb{C}\langle x_1, \ldots, x_n, x_1^*, \ldots, x_n^* \rangle \cong \mathbb{C}\langle x_1, \ldots, x_{2n} \rangle$$

one cannot assume that this remains true. So it is unlikely that Theorem 2.2.10 remains true for $\mathcal{W}_n$.

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\[\text{Not as a } *\text{-algebra.}\]
3 The geometry of free stability

In the commutative situation it was obvious what a semialgebraic set is. However, this question becomes a serious problem when we turn to the non-commutative polynomials. Not only the definition is a problem. The topology becomes a problem as well. The topology we use should maintain as many algebro-geometric relationships between quadratic modules and semialgebraic sets as possible. One suitable topology is the hull-kernel topology, which we will examine in some detail in the following section.

3.1 The hull-kernel topology

In the following let $\mathcal{A}$ be a $*$-algebra and $\mathcal{H}$ a SEPARABLE HILBERT SPACE with INNER PRODUCT $\langle \cdot, \cdot \rangle$. The aim of this section is to establish a non-commutative version of the Zariski topology called the HULL-KERNEL TOPOLOGY or sometimes the JACOBSON TOPOLOGY. In order to do so, we will need the following basic material that can be found in the second and third chapter of [18].

Definition 3.1.1. Let $\mathcal{A}$ be a $*$-algebra. An element of $a \in \mathcal{A}$ is called hermitian if $a^* = a$. The set of all hermitian elements of $\mathcal{A}$ is denoted with $\mathcal{A}_{\text{her}}$.

Definition 3.1.2. Let $\mathcal{A}$ be a $*$-algebra and $\pi$ a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$. The representation $\pi$ is said to be irreducible if $\mathcal{H} \neq 0$ and the only closed subspace of $\mathcal{H}$ which is invariant under $\pi(\mathcal{A})$ is $\mathcal{H}$ itself or $0$.

The irreducible representations $\pi$ of $\mathcal{A}$ give rise to a topology in the following way: Let $\text{Rep}(\mathcal{A})$ be the set of all representations of $\mathcal{A}$. For each representation $\pi : \mathcal{A} \to B(\mathcal{H})$ we take the kernel $\ker(\pi) \subseteq \mathcal{A}$, a two-sided ideal. Ideals that are the kernels of irreducible $*$-representation are called primitive. Let $\text{Prim}(\mathcal{A})$ denote the set of all primitive ideals. The closed sets in $\text{Prim}(\mathcal{A})$ are defined to be the sets

$$\mathcal{V}(Y) = \{ \mathfrak{P} \in \text{Prim}(\mathcal{A}) : \mathcal{I}(Y) \subseteq \mathfrak{P} \},$$

where $Y$ is some arbitrary subset of $\text{Prim}(\mathcal{A})$ and $\mathcal{I}(Y) = \bigcap_{\mathfrak{P} \in Y} \mathfrak{P}$. This topology on $\text{Prim}(\mathcal{A})$ is called the hull-kernel topology, or sometimes the Jacobson topology. The hull-kernel topology can be viewed as a non-commutative version of the Zariski-topology. Let us consider the subset $\text{IrrRep}(\mathcal{A})$ of all irreducible representations of $\text{Rep}(\mathcal{A})$ under the following equivalence relation $\sim$: Two representations $\pi : \mathcal{A} \to B(\mathcal{H})$ and $\pi' : \mathcal{A} \to B(\mathcal{H}')$ are equivalent if there is an Hilbert space isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{H}'$.
that transforms $\pi(x)$ into $\pi'(x)$ for all $x \in A$. There is a canonical map $\ker$ that assigns every element $\pi \in \text{Rep}(A)$ a two-sided ideal of $A$ by sending $\pi$ to $\ker(\pi)$. Restricting $\ker$ onto $\text{IrrRep}(A)$ give us a surjection

$$\text{IrrRep}(A)/ \sim \to \text{Prim}(A).$$

Thus, we can transfer the hull-kernel topology of $\text{Prim}(A)$ onto $\text{IrrRep}(A)/ \sim$ and therefore turning $\text{IrrRep}(A)/ \sim$ into a topological space. However, $\text{IrrRep}(A)/ \sim$ and $\text{Prim}(A)$ are not homeomorphic in general. That can only happen if and only if every pair of representations in $\text{IrrRep}(A)$ that have the same kernel are equivalent. For more details on that topic see [18, Proposition 3.1.6, p.71]. These considerations lead to the following definition:

**Definition 3.1.3.** The topology induced by $\text{Prim}(A)$ on $\text{Sp}(A) := \text{IrrRep}(A)/ \sim$ is called the **spectral topology**.

**Remark 3.1.4.** Here are some easy facts about $\text{Prim}(A)$ and $\text{Sp}(A)$:

- The space $\text{Prim}(A)$ is a $T_0$-space: Take two distinct points $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\text{Prim}(A)$. Then $\mathcal{V}(\mathcal{P}_1)$ is a closed subset such that $\mathcal{P}_2 \notin \mathcal{V}(\mathcal{P}_2)$.
- The subset $\{\mathcal{P}\}$ of $\text{Prim}(A)$ is closed if and only if $\mathcal{P}$ is maximal.
- We have the following equivalences: $\text{Sp}(A)$ is a $T_0$ space $\iff$ Any two $*$-representations $\pi, \pi' \in \text{IrrRep}(A)$ with $\ker(\pi) = \ker(\pi')$ are equivalent $\iff$ The mapping $\ker : \text{Sp}(A) \to \text{Prim}(A)$ is a homeomorphism.

In the following we will limit ourselves to the $*$-algebra $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ which has been one of our central objects so far. The reason for restricting ourselves onto separable Hilbert spaces is motivated by the following argument:

Let $T_1, \ldots, T_n$ denote the operators that give rise to a $*$-representation

$$\pi : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to B(\mathcal{H}),$$

where $\mathcal{H}$ is not a separable Hilbert space. Let $A$ be the $*$-algebra generated by the $T_1, \ldots, T_n$ and let $\mathcal{M}$ be the set of all monomials of $A$. Since $\mathcal{M}$ is countable, $\mathcal{M}x$ is countable for $x \in \mathcal{H}$ with $x \neq 0$. The Hilbert space $\mathcal{H}'$ generated by $\mathcal{M}x$ is separable and invariant under $\text{im}(\pi)$, implying that $\pi$ cannot be irreducible.

In other words the restriction onto separable spaces is enough to know everything about $\text{Prim}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ and $\text{Sp}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$.

**Definition 3.1.5.** The set of infinite matrices $M_\infty(\mathbb{C})$ with only finitely many non-zero entries is given by

$$M_\infty(\mathbb{C}) := \lim_{k \to \infty} M_k(\mathbb{C}),$$

where $\mathcal{H}$ is not a separable Hilbert space. Let $A$ be the $*$-algebra generated by the $T_1, \ldots, T_n$ and let $\mathcal{M}$ be the set of all monomials of $A$. Since $\mathcal{M}$ is countable, $\mathcal{M}x$ is countable for $x \in \mathcal{H}$ with $x \neq 0$. The Hilbert space $\mathcal{H}'$ generated by $\mathcal{M}x$ is separable and invariant under $\text{im}(\pi)$, implying that $\pi$ cannot be irreducible.

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where the direct limit is taken with respect to the sequence
\[ C \hookrightarrow M_2(C) \hookrightarrow \cdots \hookrightarrow M_k(C) \hookrightarrow \cdots \]
and embeddings \( M_k(C) \hookrightarrow M_{k+1}(C), A \mapsto \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \).

The aim of this section is to answer the question if the two spaces \( \text{Prim}(C \langle x_1, \ldots, x_n \rangle) \) and \( \text{Sp}(C \langle x_1, \ldots, x_n \rangle) \) are homeomorphic or not. For a \(*\)-algebra resp. \( C^*\)-algebra \( A \) the answer is usually no, because \( \text{Sp}(A) \) reflects the complexity of \( A \) by its very definition much better than \( \text{Prim}(A) \). However, \( C \langle x_1, \ldots, x_n \rangle \) is not a very complicated \( C^*\)-algebra and therefore it seems an interesting question to ask.

**Lemma 3.1.6.** The intersection of all the kernels of all finite \(*\)-representations \( \pi : C \langle x_1, \ldots, x_n \rangle \to M_\infty(C) \) is trivial. Furthermore, if \( Y \) is a subset of \( \text{Prim}(C \langle x_1, \ldots, x_n \rangle) \) then \( Y \) is dense in \( \text{Prim}(C \langle x_1, \ldots, x_n \rangle) \) if and only if \( I(Y) = \{0\} \).

**Proof:** The first statement is just a consequence of the Amitsur-Levitzki Theorem [36, Theorem, p. 487] and [20, Lemma 1.13]. The second statement results from the following consideration: If \( I(Y) = \{0\} \) then the case is clear. Suppose that \( Y \) is dense \( \text{Prim}(C \langle x_1, \ldots, x_n \rangle) \). If \( I(Y) \) contains \( \{0\} \) as a strict subset, then there is a finite \(*\)-representation \( \pi : C \langle x_1, \ldots, x_n \rangle \to M_\infty(C) \) such that \( \ker(\pi) \) does not contain \( I(Y) \). By Artin-Wedderburn \( \pi \) decomposes into a direct sum of irreducible \(*\)-representations. Thus there must be a finite irreducible \(*\)-representation \( \pi' \) such that \( \ker(\pi') \not\subset I(Y) \), which contradicts our assumption that \( Y \) is dense.

**Lemma 3.1.7.** There is a monotone increasing sequence \( (d_j)_j \) in \( \mathbb{N} \) and a sequence of representations \( \pi_j : \mathbb{C}[x] \to M_\infty(\mathbb{C}) \) such that
- for every matrix \( A \in \text{Her}_{d_j} \), there is a \( f \in \mathbb{R}[x] \) with \( \pi_j(f) \sim A \), where \( \sim \), denotes similarity.
- we have \( \bigcap_j \ker(\pi_j) = \{0\} \).

**Proof:** The proof is divided up into two steps:

**Step I:** Let the \(*\)-representation \( \pi_j \) be given by a matrix \( T \in \text{Her}_{d_j} \) with eigenvalues \( 0 < \lambda_{d_j} < \ldots < \lambda_1 \leq 1 \).

Let \( T' \in \text{Her}_{d_j} \) be another matrix with eigenvalues \( \lambda'_1, \ldots, \lambda'_m \) for some \( m \leq d_j \). Take one polynomial \( f \in \mathbb{R}[x] \) such that \( f \) maps the eigenvalues of \( T \) onto the eigenvalues of \( T' \). By the Spectral Mapping Theorem we have

\[ \sigma(\pi(f)) = \sigma(T'), \]

implying that \( \pi(f) \) and \( T' \) are similar.

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Step II: Let $0 < \lambda_{d_j} < \cdots < \lambda_1 \leq 1$ for $j \in \mathbb{N}$ and suppose that $T_j$ converges in $B(\mathcal{H})$. The limit of the $*$-representations $\pi_j$ gives rise to a $*$-representation $\pi : \mathbb{C}[x] \to B(\mathcal{H})$. By using the spectral mapping theorem we see that $\sigma(\pi(f)) \neq \{0\}$, whenever $f \neq 0$. Thus, $\pi(f) = 0$ if and only if $f = 0$ and since $\ker(\pi) = \bigcap_j \ker(\pi_j)$, we are done. □

In the following we will study the topologies of $\text{Prim}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ and $\text{Sp}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$. The question that we are going to answer is if $\text{Prim}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ is homeomorphical to $\text{Sp}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ with respect to the $\ker$ mapping.

**Lemma 3.1.8.** For $k \in \mathbb{N}$ let $\text{Sp}_k(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ be the subset of $\text{Sp}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ that consists of all $*$-representations of dimension $\leq k$.

Then $\text{Sp}_k(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ and $\text{Sp}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)/\mathcal{I}(\text{Sp}_k(\mathbb{C}\langle x_1, \ldots, x_n \rangle))$ are homeomorphic.

**Proof:** See [13] Proposition 3.6.3, p. 85]. □

**Example 3.1.9.** Let us consider the case above for $k = 1$. Every irreducible representation is given by a point in $\mathbb{C}^n$. Thus, $\mathcal{I}(\ker(\text{Sp}_1(\mathbb{C}\langle x_1, \ldots, x_n \rangle))) = 3$.

Since $\mathbb{C}\langle x_1, \ldots, x_n \rangle/3 \cong \mathbb{C}[x_1, \ldots, x_n]$, we get that $\text{Sp}_1(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ is homeomorphic to the set of closed points of $\text{Spec}(\mathbb{C}[x_1, \ldots, x_n])$.

But $\text{Specm}(\mathbb{C}[x_1, \ldots, x_n])$ is nothing more than $\text{Prim}(\mathbb{C}[x_1, \ldots, x_n])$ which is again homeomorphic to the closed set $\mathcal{V}(\ker(\text{Sp}_1(\mathbb{C}\langle x_1, \ldots, x_n \rangle)))$ in $\text{Prim}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$. Thus, we have the following diagram of homeomorphisms:

\[
\begin{array}{ccc}
\text{Sp}_1(\mathbb{C}\langle x_1, \ldots, x_n \rangle) & \cong & \text{Sp}(\mathbb{C}\langle x_1, \ldots, x_n \rangle)/3 \\
\mathcal{V}(\ker(\text{Sp}_1(\mathbb{C}\langle x_1, \ldots, x_n \rangle))) & \cong & \text{Prim}(\mathbb{C}[x_1, \ldots, x_n])
\end{array}
\]

**Definition 3.1.10.** We say that an operator $A \in B(\mathcal{H})$ has a **Matrix Representation** if there is an infinite matrix $(a_{ij})_{ij}$ with $a_{ij} \in \mathbb{C}$ such that for every $x = \sum_k b_k e_k$ we have $Ax = \sum_j c_j e_j$, where $(e_i)_i$ is an orthonormal basis of $\mathcal{H}$ and $c_j = \sum_k a_{jk} b_k$.

**Lemma 3.1.11.** Let $A$ be some operator on a separable Hilbert space $\mathcal{H}$. If $A$ admits a matrix representation with respect to some orthogonal basis, then $A$ is bounded. If an infinite matrix $(a_{ij})_{ij}$ satisfies the inequality

\[
\left| \sum_{i=1}^p \sum_{j=1}^q a_{ij} x_i y_j \right| \leq M \left| \sum_{i=1}^p |x_i|^2 \right|^{1/2} \left| \sum_{j=1}^q |y_j|^2 \right|^{1/2},
\]

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for some $M > 0$ and any numbers $p, q \in \mathbb{N}$, $x_1, \ldots, x_p, y_1, \ldots, y_q \in \mathbb{C}$, then $(a_{ij})_{ij}$ defines a bounded operator.

**Proof:** See [5 Theorem p.50]+[5 Theorem p.53].

**Remark 3.1.12.** By using Lemma 3.1.7 and Lemma 3.1.11 we can construct a $*$-representation $\pi : \mathbb{C}[x] \to B(\mathcal{H})$ with the property that $\ker(\pi) = \{0\}$. By taking a self-adjoint non-compact bounded operator with an infinite spectrum, then induced $*$-representation $\pi' : \mathbb{C}[x] \to B(\mathcal{H})$ satisfies $\ker(\pi') = \{0\}$ by the spectral mapping theorem. Now $\pi$ and $\pi'$ cannot be equivalent. This suggests that $\text{Prim}(\mathbb{C}[x_1, \ldots, x_n])$ and $\text{Sp}(\mathbb{C}(x_1, \ldots, x_n))$ are not homeomorphic. However, $\pi$ and $\pi'$ do not need to be irreducible. In order to verify the assertion we therefore turn to the ring $\mathbb{C}(x_1, x_2)$:

We identify the separable Hilbert space $\mathcal{H}$ with $\ell^2(\mathbb{C})$ together with an orthonormal basis $(e_n)_n$ and consider the right shift operator $R : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$ resp. left shift operator $L : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C})$. The operators $R' = \frac{1}{2}(R + L)$ and $L' = \frac{1}{2}(R - L)$ are both contained in $B(\mathcal{H})_{\ker}$. The algebra $\mathcal{A}$ generated by $R'$ and $L'$ equals $\mathcal{A}'$, the algebra generated by $R$ and $L$. This follows from $R' + iL' = R$ and $R' + \frac{i}{2}L' = L$. Thus we can restrict ourselves to $\mathcal{A}'$. Take $x \in \ell^2(\mathbb{C})$ with $x_n \neq 0$ for some $n \in \mathbb{N}$. Then

$$
(R^nL^n - R^{j+1}L^{n+1})x \in \mathbb{C}e_j,
$$

implying $e_j \in \mathcal{A}'x$ for $j \in \mathbb{N}$. It is easy to see that $e_n \in \mathcal{A}'x$ also means that the all the orthonormal basis vectors $e_1, \ldots, e_n, \ldots$, must be contained in $\mathcal{A}'x$. The Hilbert space generated by $\mathcal{A}'x$ must be the whole space $\ell^2(\mathbb{C})$. Hence, the $*$-representation given by $R'$ and $L'$ is irreducible. Furthermore, the kernel of this representation is trivial: It is enough to show that for each element $f \in \mathcal{A}'$ that $\exists x \in \ell^2(\mathbb{C}) : f x \neq 0$. Since $LR = \text{id}_{\ell^2(\mathbb{C})}$, we can assume that $f = \sum a_\alpha R^{\alpha_1}L^{\alpha_2}$. There is a point $y \in \ell^2(\mathbb{C})$ such that $(\sum_a a_\alpha R^{\alpha_1}L^{\alpha_2})y \neq 0$. Let $k = \max[|\alpha_1| + |\alpha_2| : a_\alpha \neq 0]$ and

$$
y = (1, \ldots, 1, 0, \ldots).
$$

Let $\alpha'$ be such that $\alpha' \in \min\{\alpha_2 : a_\alpha \neq 0\}$ and $\alpha'_1 = \min\{(\alpha_1, \alpha'_2) : a_{(\alpha_1, \alpha'_2)} \neq 0\}$. Then $R^{\alpha'_1}L^{\alpha_2'}y \neq 0$ and $R^{\alpha'_1}L^{\alpha_2'}y \neq -\sum_{\alpha \neq \alpha'} a_\alpha R^{\alpha_1}L^{\alpha_2}y$.

Let us now consider the modified shift operators

$$\hat{R} : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C}), (a_1, \ldots, a_n, \ldots) \mapsto \left(0, a_1, \frac{\exp(-2)}{2}a_2, \ldots, \frac{\exp(-n)}{n}a_n, \ldots\right),$$

resp.

$$\hat{L} : \ell^2(\mathbb{C}) \to \ell^2(\mathbb{C}), (a_1, \ldots, a_n, \ldots) \mapsto \left(a_2, \frac{\exp(-2)}{2}a_3, \ldots, \frac{\exp(-n+1)}{n-1}a_n, \ldots\right).$$
Again we have $\frac{1}{2}(\hat{R} + \hat{L}), \frac{1}{2}(\hat{R} - \hat{L}) \in K(\mathcal{H})_{\text{her}}$ and as before we just need to consider the algebra $\mathcal{B}$ generated by $\hat{R}$ and $\hat{L}$. Take $x \in \ell^2(\mathbb{C})$ with $x_n \neq 0$. For $j \geq 2$ we have

$$\hat{R}^j \hat{L}^n x - \hat{R}^{j+1} \hat{L}^{n+1} x = \begin{cases} 0 & \text{for } k < j \\ \beta_{n,r} x_{n+r} & \text{for } k = j \\ \beta_{n,r} x_{n+r} & \text{for } k = j + r, r > 0 \end{cases}$$

where

$$\beta_{n,1} = \left( \prod_{m=0}^{n-2} \frac{\exp(-n + m)}{n - m} \right) \prod_{m=2}^{j} \frac{\exp(1 - m)}{m} - \left( \prod_{m=0}^{n-2} \frac{\exp(-n + m)}{n - m} \right) \prod_{m=2}^{j} \frac{\exp(1 - m)}{m} = 0$$

and

$$\beta_{n,r} = \left( \prod_{m=1}^{n} \frac{\exp(-n - r + m)}{n + r - m} \right) \prod_{m=0}^{j} \frac{\exp(-r - m)}{r + m} - \left( \prod_{m=1}^{n} \frac{\exp(-n - r + m)}{n + r - m} \right) \prod_{m=0}^{j+1} \frac{\exp(-r - m)}{r + m}$$

for $r > 2$. There are two cases that we need to consider:

- **Case I:** The sequence $x$ has only a finite amount of elements that do not vanish. In that situation there is a number $n \in \mathbb{N}$ such that $x_n \neq 0$ and $x_k = 0$ for all $k > n$. Now

$$\hat{L}^n x = \left( \prod_{m=1}^{n-2} \frac{\exp(-n + m)}{n - m} x_n, 0, \ldots, 0, \ldots \right)$$

and therefore $e_1 \in \mathcal{B}a$, implying that $e_1, \ldots, e_n, \ldots \in \mathcal{B}x$. Thus, the closure of $\mathcal{B}x$ is the whole space $\ell^2(\mathbb{C})$.

- **Case II:** The sequence $x$ has an infinite amount of elements that do not vanish. Take a natural number $n \in \mathbb{N}$ such that $\frac{2^{n+1}}{n^{n+1}} < 1$ for all $n' \geq n$. Consider the sequence $(b_n)_n \subseteq \ell^2(\mathbb{C})$ with $b_n = \hat{R}^j \hat{L}^n x - \hat{R}^{j+1} \hat{L}^{n+1} x$ for some fixed $j \geq 2$. We have $\beta_{n,r} > 0$ resp.

$$\left( \prod_{m=1}^{n} \exp(-n - r + m) \right) \left( \prod_{m=0}^{j} \exp(-r - m) \right) \geq \left( \prod_{m=1}^{n+1} \exp(-n - r + m) \right) \left( \prod_{m=0}^{j+1} \exp(-r - m) \right)$$

for big $n \in \mathbb{N}$. This leads to

$$\frac{\beta_{n,r}}{\prod_{m=1}^{n-2} \frac{\exp(-n + m)}{n - m} \prod_{m=2}^{j+1} \frac{\exp(-r - m)}{r + m}} \leq \frac{\prod_{m=1}^{n+1} \exp(-n - r + m) \prod_{m=0}^{j+1} \exp(-r - m)}{\prod_{m=1}^{n} \frac{\exp(-n - r + m)}{n + r - m} \prod_{m=2}^{j+1} \frac{\exp(-r - m)}{r + m}}$$
Set $\alpha_j = \prod_{m=0}^{j} \frac{\exp(-r_m \cdot \lambda)}{\prod_{m=1}^{j} \exp(r_m \cdot \lambda)}$. With [3.1] we get

$$
\| \frac{b_n}{(\prod_{m=1}^{n-2} \exp(-m \cdot \lambda) / (n-m)} x_n - e_j \|_{L^2(C)}^2 \leq \sum_{r=1}^{\infty} \left( \frac{\prod_{m=1}^{n-2} \exp(-r \cdot \lambda)}{\prod_{m=1}^{n} \exp(-m \cdot \lambda) / (n-m)} \right) \alpha_j^2 \left( \frac{\exp(-r \cdot \lambda)}{\exp(r \cdot \lambda)} \right) \alpha_j^2
$$

$$
= \sum_{r=1}^{\infty} \left( \exp(-r \cdot \lambda) \exp(r \cdot \lambda) \prod_{m=1}^{n-2} \frac{-(n + m)}{-(n - r + m)} \right) \alpha_j^2 \leq \sum_{r=1}^{\infty} \left( \exp(-r \cdot \lambda) \exp(r \cdot \lambda) \right) \alpha_j^2
$$

$$
= \frac{\alpha_j^2}{\exp(2n - 4) - 1}
$$

Thus, the sequence $(b_n)_n$ converges towards $e_j$ in the $L^2(C)$ norm for $n \to \infty$. In other words, $e_j$ is arbitrary close to $Bx$ for $j \geq 2$. It is not difficult to see that this is also the case for $e_1$, implying that the closure of $Bx$ is the whole space $L^2(C)$.

The two cases above now imply that the operators $\frac{1}{2}(\tilde{R} + \tilde{L})$ and $\frac{1}{2}(\tilde{R} - \tilde{L})$ give rise to an irreducible $*$-representation $\tilde{\pi}$ with $\text{im}(\tilde{\pi}) \subseteq K(L^2(C))$. It remains to verify that $\tilde{\pi}$ has a trivial kernel. In contrast to the other case we have $\tilde{L} \tilde{R} \neq \text{id}_{L^2(C)}$. For an element $x \in L^2(C)$ we define

$$
N(x) = \{ n \in \mathbb{N} : x_n \neq 0 \}.
$$

Obviously $N(\tilde{L} \tilde{R} x) = N(x)$. Using $N(x)$ we can define an equivalence relation on $L^2(C)$: Two elements $x, x' \in L^2(C)$ are equivalent if and only if $N(x) = N(x')$. With respect to that relation it is not difficult to see that we can use the same arguments we used with respect to $R$ and $L$ to prove that $\tilde{\pi}$ has a trivial kernel.

An immediate consequence of Remark 3.1.12 is:

**Proposition 3.1.13.** The mapping $\ker : \text{Sp}(\mathbb{C}(x_1, \ldots, x_n)) \to \text{Prim}(\mathbb{C}(x_1, \ldots, x_n))$ is not a homeomorphism for $n \geq 2$.

**Proof:** In Remark 3.1.12 we saw that the operators $R' = \frac{1}{2}(R + L)$ and $L' = \frac{1}{2}(R - L)$ gave rise to an irreducible $*$-representation $\pi$ with trivial kernel. The compactifications $\tilde{R}$ and $\tilde{L}$ of $R'$ and $L'$ also gave rise to an irreducible $*$-representation $\tilde{\pi}$ with trivial kernel. Let $f \in \mathbb{C}(x_1, x_2)$ be a polynomial with no constant term. The image of $\pi(f)$ contains bounded operators that are not compact, while the image of $\tilde{\pi}(f)$ just contains compact operators. No isomorphism $\mathcal{H} \rightarrowtail \mathcal{H}$ is able to transform $\tilde{\pi}(f)$ into $\pi(f)$. □

**Remark 3.1.14.** According to Lemma 3.1.11 the left and right shift resp. the compactified left and right shift operators have a representation as infinite matrices. If we consider the finite dimensional versions of $R'$ and $L'$ along with their compactified versions, then they are still not unitary equivalent.
3.2 Hull-kernel topology and semialgebraic sets

In the previous section we established the basics of the hull-kernel topology. The next step is to consider certain semi-algebraic sets given by the generators of a quadratic module and investigate how stability is encoded in the geometry of these semialgebraic sets. Here the hull-kernel topology will play an important role.

The most important question is, however, which framework should we use? One answer would be the $C^*$-algebra of compact operators over a Hilbert space $\mathcal{H}$. Since compact operators have the approximation property, i.e., they can be approximated by projections, they seem to be the obvious choice. In order to progress further we will need to define what a positive-definite operator is. Considering the finite dimensional space $M_n(\mathbb{C})$ one property we want for our space is the following equivalence: $T$ is positive definite $\iff \sigma(T) \subseteq (0, \infty)$. In other words, the definition of a positive definite operator should be in such a way that the spectrum is contained in $(0, \infty)$. One approach is the definition of a strictly positive element of a $C^*$-algebra $A$. An element $t$ is called strictly positive if $\langle t, \varphi \rangle > 0$ for all states $\varphi \in \text{St}(A)$ of $A$. But if $A$ has no unit, the desired equivalence is impossible: For example, any non-negative operator $T \in K(\mathcal{H})$ with infinite dimensional range satisfies $0 \in \sigma(T)$. In contrast to the $C^*$-algebra $K(\mathcal{H})$, the $C^*$-algebra $B(\mathcal{H})$ is unital and has the property of being strictly positive in the sense we wanted. But bounded operators have again some disadvantages. Not every bounded operator can be approximated by finite dimensional range operators. The solution to this problem is to consider the unitarization

$$K^{un}(\mathcal{H})$$

of $K(\mathcal{H})$. The underlying vector space of $K^{un}(\mathcal{H})$ is $K(\mathcal{H}) \oplus \mathbb{C}$, the multiplication is given by

$$(T_1 \oplus z_1)(T_2 \oplus z_2) = (T_1 T_2 + z_1 T_2 + z_2 T_1) \oplus (z_1 z_2),$$

the involution $*$ by

$$(T \oplus z)^* = T^* \oplus \overline{z},$$

and the norm of $T \oplus z$ is defined to be

$$\sup_{T' \in K(\mathcal{H}), \|T'\| \leq 1} \|TT' + zT'\|.$$

**Definition 3.2.1.** Let $A$ be an unital $C^*$-algebra. We say that $f \in A_{\text{her}}$ is non-negative $f \succeq 0$ if $\varphi(f) \geq 0$ for all $\varphi \in \text{St}(A)$ and positive $f \succ 0$ if $\varphi(f) > 0$ for all $\varphi \in \text{St}(A)$.

**Lemma 3.2.2.** Let $A$ be an unital $C^*$-algebra, $a \in A$ with $a \succeq 0$ and $\pi \in \text{Rep}(A)$ non-degenerated\(^1\) Consider the statements:

- a The element $a \in A$ is positive.

\(^1\)Non-degenerated means that for every $x \in \mathcal{H}\setminus\{0\}$ we can find an element $a \in A$ such that $\pi(a)x \neq 0$. 

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b We have \( \overline{\pi(\mathcal{A})} = \mathcal{H} \).

Then statement (a) implies statement (b).

**Proof:** See [1] Lemma 1, p. 749. \( \square \)

We will use Lemma 3.2.2 to show that Definition 3.2.1 defines positivity in a ‘proper’ way.

**Proposition 3.2.3.** Let \( \mathcal{H} \) be a Hilbert-space and \( T : \mathcal{H} \to \mathcal{H} \) a operator in \( B(\mathcal{H})_{\text{her}} \). Then the following statements hold:

(i) If \( T \) is non-negative, then the spectrum \( \sigma(T) \) of \( T \) is contained in \([0, \infty)\).

(ii) If \( T \) is positive, then the spectrum \( \sigma(T) \) of \( T \) is contained in \((0, \infty)\).

**Proof:** (i): This is [55] Lemma VI.4.4, p. 278.

(ii): We have to verify that \( 0 \in \mathcal{g}(T) \), i.e that \( T \) is invertible. The identity \( B(\mathcal{H}) \to B(\mathcal{H}) \) is a non-degenerated representation of \( B(\mathcal{H}) \) and so by Lemma 3.2.2 we get that \( TB(\mathcal{H}) \) is dense in \( B(\mathcal{H}) \). Since \( TB(\mathcal{H}) \) is dense in \( B(\mathcal{H}) \), so is \( TB(\mathcal{H})T \). Thus there exists an element \( X \in B(\mathcal{H}) \) such that \( \|TXT - \text{id}_H\| < \frac{1}{2} \). Using the Neumann series we see that \( TXT \) is invertible. Thus, \( T \) has a right and a left inverse and therefore \( T \) is invertible. \( \square \)

**Remark 3.2.4.** Note that Proposition 3.2.3 also holds true if we replace \( B(\mathcal{H})_{\text{her}} \) with \( K^\text{un}_\text{her}(\mathcal{H}) \). In fact Proposition 3.2.3 can be generalized to work for all unital \( C^* \)-algebras. Using this we can show that the two sets \( U_1 = \{T \in B(\mathcal{H})_{\text{her}} : T > 0\} \) and \( U_2 = \{T \in K^\text{un}_\text{her}(\mathcal{H}) : T > 0\} \) are open in \( B(\mathcal{H})_{\text{her}} \) resp. \( K^\text{un}_\text{her}(\mathcal{H}) \).

Let us start with \( U_1 \). Take a point \( T \in U_1 \) and let \( B \in B(\mathcal{H})_{\text{her}} \). Then by [30] Theorem 4.10, p. 291 we have that \( T + B \) is again self-adjoint with \( \text{dist}(\sigma(T + B), \sigma(T)) \leq \|B\| \). Since \( \sigma(T) \subseteq (0, \infty) \), we see that \( \sigma(T + B) \) will be contained in \((0, \infty)\) for small \( \|B\| \).

By Proposition 3.2.3 we are done.

Let us turn our attention to the set \( U_2 \). The dual space \( K^\text{un}(\mathcal{H})^\vee \) of \( K^\text{un}(\mathcal{H}) \) is equal to \( K(\mathcal{H})^\vee \ominus \mathbb{C} \). A state of \( K^\text{un}(\mathcal{H}) \) can be viewed as a state of \( B(\mathcal{H}) \), since any state of \( K^\text{un}(\mathcal{H}) \) can be extended to a state of \( B(\mathcal{H}) \) by the HAHN-BANACH THEOREM. An element \( T \ominus z \in K^\text{un}(\mathcal{H}) \) is positive if and only if \( T + z\text{id}_H \) is positive for all states of \( B(\mathcal{H}) \) that come from \( K^\text{un}(\mathcal{H}) \). Since the spectrum of a positive element \( T \ominus z \) is a compact subset of \((0, \infty)\) and behaves well under small perturbations, we see that \( T \ominus z \) is also an inner point of \( U_2 \). Thus, \( U_2 \) is open as well.

**Definition 3.2.5.** A basically closed semi-algebraic set \( S_{\text{nc}}^\text{un} \) in \( K^\text{un}_\text{her}(\mathcal{H})^n \) is given by a finite number of polynomials \( f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}} \) with

\[
S_{\text{nc}}^+ := S^+(f_1, \ldots, f_r) = \{(T_1, \ldots, T_n) \in K^\text{un}_\text{her}(\mathcal{H})^n : f_1(T_1, \ldots, T_n), \ldots, f_r(T_1, \ldots, T_n) \geq 0\}.
\]
Replacing $\succeq$ with $\succ$ in the definition of $S^{nc}$, will be denoted with $(S^{nc+})^\circ$. In the same manner, we define a basically semi-algebraic set $S^{nc}$ in $K(H)_{\text{her}}$ resp. $B(H)_{\text{her}}$ by

$$S^{nc} := S(f_1, \ldots, f_r) = \{(T_1, \ldots, T_n) \in K(H)_{\text{her}}^n : f_1(T_1, \ldots, T_n), \ldots, f_r(T_1, \ldots, T_n) \geq 0\},$$

resp.

$$S^{nc}_{B} := S_B(f_1, \ldots, f_r) = \{(T_1, \ldots, T_n) \in B(H)_{\text{her}}^n : f_1(T_1, \ldots, T_n), \ldots, f_r(T_1, \ldots, T_n) \geq 0\}.$$

Replacing $\succeq$ with $\succ$ in the definition, will be denoted with $(S^{nc+})^\circ$, $S^{nc}$ and $S^{nc+}_{B}$.

Considering compact operators and semi-algebraic sets of them has a huge advantage. Hilbert spaces satisfy the approximation property, which means that every compact operator can be approximated by a sequence of finite rank operators. Therefore, if we have given a semi-algebraic set $S$ it makes sense to see the finite dimensional slices of this set and investigate what information about $S$ they contain. These thoughts lead us to the following

**Definition 3.2.6.** Let $V \subseteq B(H)^n$ be a subset. The $k$-th level $V_k$ of $V$ is defined in the following way:

Let $G(H)$ the Grassmanian of $H$ and $G_k(H) \subseteq G(H)$ the subset of all $k$-dimensional subspaces. A tuple $(T_1, \ldots, T_n)$ is contained in $V_k$ if and only if it fits into the commutative diagram

$$
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
(P_1, \ldots, P) \\
(P_1, \ldots, P) \\
(P_1, \ldots, P) \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
\begin{array}{c}
\mathcal{H}^n \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{H}^n \\
\end{array}
$$

for some $(T_1, \ldots, T_n) \in V$, $\mathcal{H}^n \subseteq G_k(H)$ and orthogonal projection $P : \mathcal{H} \to \mathcal{H}'$.

**Proposition 3.2.7.** Let $f_1, \ldots, f_r \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$ be polynomials with

$$(S^{nc+})^\circ = (S^+)^\circ \langle f_1, \ldots, f_r \rangle.$$

Suppose that $(S^{nc+})^\circ \neq \emptyset$. Then we have int$((S^{nc+})^\circ) = (S^{nc+})^\circ$ and that there exists a natural number $k$ such that $(S^{nc+})^\circ_k \neq \emptyset$.

**Proof:** Step I: int$((S^{nc+})^\circ) = (S^{nc+})^\circ$. We have to show that $(S^{nc+})^\circ$ is open. Each $f_i$ for $i = 1, \ldots, r$ can be interpreted as a continuous function

$$f_i : K_{\text{her}}(\mathcal{H})^n \to K_{\text{her}}(\mathcal{H})$$

and $(S^{nc+})^\circ$ is nothing more than the intersection of the preimages $f_i^{-1}(\{T \in K_{\text{her}}(\mathcal{H}) : T > 0\})$ for $i = 1, \ldots, r$. By Remark 3.2.4, we know that

$$f_i^{-1}(\{T \in K_{\text{her}}(\mathcal{H}) : T > 0\})$$
is open for $i = 1, \ldots, r$ and therefore $(S^{nc^+})^c$ is open.

Step II: For $k \in \mathbb{N}$ big enough we have $(S^{nc^+})^c_k \neq \emptyset$: Write $T_i = T'_i \oplus z_i$ for $T'_i \in (S^{nc^+})^c$ and $i = 1, \ldots, n$. Each of the compact operators $T'_i$ can be approximated by a sequence $(T'_{ik})_k \subseteq \mathcal{M}_\infty(\mathbb{C})$. Thus,

$$
\begin{align*}
    f_1(T'_{1k} \oplus z_1, \ldots, T'_{nk} \oplus z_n) \\
    \vdots \\
    f_r(T'_{1k} \oplus z_1, \ldots, T'_{nk} \oplus z_n)
\end{align*}
$$

converge towards

$$
\begin{align*}
    f_1(T_1, \ldots, T_n) \\
    \vdots \\
    f_r(T_1, \ldots, T_n).
\end{align*}
$$

But $\text{int}((S^{nc^+})^c) = (S^{nc^+})^c \neq \emptyset$ implies that $(T'_{1k} \oplus z_1, \ldots, T'_{nk} \oplus z_n)$ will end up in $(S^{nc^+})^c$ for a large enough length index $k$, which proves the assertion. \hfill \Box

Remark 3.2.8. Let us examine the geometric implications of Proposition 3.2.7 with respect to compact operators. Consider the set

$$
S^{nc} = S^{nc}(x) = \{T \in K(\mathcal{H})_{\text{her}} : T \succeq 0\}.
$$

For each $k \in \mathbb{N}$ we have $S^{nc}_k = S^{nc}_B$. However, any $T \in S^{nc}$ cannot be an inner point, since $0 \notin \partial(T)$. Thus, we have $S^{nc}_B \neq S^{nc}$. Turning to the case where

$$
S^{nc}_B = S^{nc}(x) = \{T \in B(\mathcal{H})_{\text{her}} : T \succeq 0\}
$$

we see with Proposition 3.2.3 that $S^{nc}_B = S^{nc}_B$ and how much more information $S^{nc}_B$ contains than $S^{nc}$. The corresponding sets $S^{nc^+}$ and $(S^{nc^+})^c$ can be thought as a synthesis.

The considerations in Remark 3.2.8 lead us to the following two statements:

Proposition 3.2.9. Let $f_1, \ldots, f_r \in \mathcal{C}(x_1, \ldots, x_n)_{\text{her}}$ be polynomials with

$$
S^{nc^+}_B = S^{nc^+}_B(f_1, \ldots, f_r).
$$

Suppose that $S^{nc^+}_B \neq \emptyset$. Then we have $\text{int}(S^{nc^+}_B) = S^+_B$.

Proof: Follows directly from Proposition 3.2.3 \hfill \Box

Let us consider a finitely generated quadratic module $M^{nc} = QM(f_1, \ldots, f_r)$. Then $M^{nc}$ gives rise to the subset

$$
\widetilde{M}^{nc} := \{\pi \in \text{Rep}(\mathbb{C}(x_1, \ldots, x_n)) : \forall f \in M^{nc} : \pi(f) \geq 0\}
$$
of $\Rep(C(x_1, \ldots, x_n))$. More generally, for any subset $F \subseteq \C(x_1, \ldots, x_n)$ we define the set
\[
\tilde{\mathcal{F}} := \{ \pi \in \Rep(\C(x_1, \ldots, x_n)) : \forall f \in F : \pi(f) \geq 0 \}
\]
and similarly
\[
\tilde{\mathcal{F}}_k := F \cap \Rep_k(\C(x_1, \ldots, x_n))
\]
The main advantage of the set $\tilde{\mathcal{F}}$ is that we have access to the hull-kernel topology and spectral topology which we introduced earlier: We say that $\tilde{\mathcal{F}}$ is closed with respect to the spectral topology if
\[
\tilde{\mathcal{F}} \cap \Irr\Rep(\C(x_1, \ldots, x_n))
\]
is closed in $\Sp(\C(x_1, \ldots, x_n))$. Similarly, $\tilde{\mathcal{F}}$ is closed with respect to the hull-kernel topology if the set
\[
\ker(\tilde{\mathcal{F}} \cap \Irr\Rep(\C(x_1, \ldots, x_n)))
\]
is closed in $\Prim(\C(x_1, \ldots, x_n))$ and so on.

The situation is somewhat similar to the correspondence of semialgebraic sets in $\R^n$ and the constructible subsets in $\Spec(\R[x_1, \ldots, x_n])$ which is a subset of the real affine space $\Spec(\R[x_1, \ldots, x_n])$.

For $F = \{ f_1, \ldots, f_r \} \subseteq \C(x_1, \ldots, x_n)$ we can identify the semialgebraic set $S_{nc}^F$ with $\tilde{\mathcal{F}}$ in the obvious way.

By identifying a subset of $V \subseteq S_{nc}^F$ with the corresponding one in $\tilde{\mathcal{F}}$, we can also talk about the hull-kernel resp. spectral topology of subsets of $S_{nc}^F$.

**Definition 3.2.10.** Let $V \subseteq S_{nc}^F$ and consider the identification
\[
E(\mathcal{H})^n \leftrightarrow \Rep(\C(x_1, \ldots, x_n)).
\]
Thus, we can view $V$ as a subset of $\Rep(\C(x_1, \ldots, x_n))$. We say that $V$ is dense with respect to the hull-kernel topology if
\[
\mathcal{V}(\mathcal{I}(V)) = \Prim(\C(x_1, \ldots, x_n)).
\]
By using the mapping $\ker : \Sp(\C(x_1, \ldots, x_n)) \to \Prim(\C(x_1, \ldots, x_n))$ we also have a notion of denseness with respect to the spectral topology.

**Lemma 3.2.11.** Let $S_{nc}^F := S(\{ f_1, \ldots, f_r \})$ be a semi-algebraic set in $K(\mathcal{H})^n_{nc}$. If $\text{int}(S_{nc}^F_k) \neq \emptyset$ for big $k \in \N$, then $S_{nc}^F$ is dense in both, the hull-kernel topology and the spectral topology.

\[\text{For more information on that matter see [8, Proposition 7.2.2, p. 143]}
\[\text{3}\mathcal{I}(V) = \bigcap_{\pi \in \mathcal{V}} \ker(\pi).
\]

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Remark 3.2.12. Lemma 3.2.11 works also for \( ~ \)-representations. So we get By Artin-Wedderburn each \( T^{(0)} = \cdots T^{(n)} \) such that (\( T^{(0)}, \ldots, T^{(n)} \)) for some \( t \in \mathbb{R} \). The polynomial \( f(tT^{(0)}, \ldots, tT^{(n)}) = M_n(\mathbb{C}[t]) \) vanishes on some interval with non-empty interior and therefore it must vanish everywhere. Since the point \( (T^{(0)}, \ldots, T^{(n)}) \) was arbitrary, we conclude that \( f \) must vanish on the whole space \( \text{Her}^k \). In other words, \( f \) must be in the kernel of all finite \( * \)-representations \( \pi \in \text{Rep}(\mathbb{C}(x_1, \ldots, x_n)) \) which is \( \{0\} \) according to Lemma 3.1.6. Let \( F = \{f_1, \ldots, f_r\} \).

The above statement implies that

\[
\bigcap_{k} \bigcap_{\pi \in \mathcal{F}_k} \ker(\pi) = \{0\}.
\]

By Artin-Wedderburn each \( \pi \in \mathcal{F}_k \) is a finite direct sum of finite dimensional irreducible \( * \)-representations. So we get

\[
\bigcap_{k} \bigcap_{\pi \in \mathcal{F}_k \cap \text{IrrRep}_k(\mathbb{C}(x_1, \ldots, x_n))} \ker(\pi) = \{0\},
\]

i.e. \( \mathcal{F} \) resp. \( S^{\text{ac}} \) is dense with respect to the hull-kernel and spectral topology.

Remark 3.2.12. Lemma 3.2.11 works also for \( S^{\text{ac}}_B = S^{\text{ac}}_B(f_1, \ldots, f_r) \) if \( \text{int}(S_B) \neq \emptyset \) and \( P_k S^{\text{ac}}_B P_k \subseteq S^{\text{ac}}_B \) for any orthogonal projection \( P_k \) onto a \( k \)-dimensional subspace of \( \mathcal{H} \). The proof is just Proposition 3.2.9 combined with the arguments of Lemma 3.2.11.

Remark 3.2.13. Let \( V \) be a variety in \( \mathbb{C}^n \). Then \( V \) is Zariski-dense if and only if \( V = \mathbb{C}^n \). The situation with respect to the hull-kernel topology is much different as the following example will show. To be more precise, there are subsets \( V \) of \( \mathcal{K}(\mathcal{H})^n \) with the following properties:

- The set \( V \) has empty interior with respect to the norm topology in \( \mathcal{K}(\mathcal{H})^n \) but is dense with respect to the hull-kernel topology.

- For big \( k \in \mathbb{N} \) the set \( V_k \neq \emptyset \) is a lower-dimensional variety.

For a polynomial \( f \in \mathbb{C}(x_1, \ldots, x_n) \) and a number \( k \in \mathbb{N} \) we define \( \text{Sing}_k(f) \) to be the set

\[
\text{Sing}_k(f) = \{(A_1, \ldots, A_n) \in M_k(\mathbb{C})^n : \det(f(A_1, \ldots, A_n)) = 0\}
\]

and \( \text{Sing}(f) \) to be the set

\[
\text{Sing}(f) = \bigcup_{k \in \mathbb{N}} \text{Sing}_k(f).
\]

Let \( f \) be an atom that does not vanish on \( \text{Sing}(f) \). Let \( g \in \mathbb{C}(x_1, \ldots, x_n) \) be a polynomial that vanishes on \( \text{Sing}(f) \). Thus \( \text{Sing}(f) \subseteq \text{Sing}(g) \). We will show that \( g = 0 \).

\[
\text{By [16] a non-zero non-unit of } \mathbb{C}(x_1, \ldots, x_n) \text{ is an atom if it cannot be written as a product of two non-units.}
\]

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Suppose the contrary, i.e. \( g \neq 0 \). By [28, Theorem 2.12, p.11] we get that \( f \) is an atomic factor of \( g \) up to STABLE ASSOCIATIVITY. Stably associativity of two polynomials \( h_1 \) and \( h_2 \) in \( \mathbb{C}(x_1, \ldots, x_n) \) just means that
\[
\mathbb{C}(x_1, \ldots, x_n)/h_1 \mathbb{C}(x_1, \ldots, x_n) \cong \mathbb{C}(x_1, \ldots, x_n)/h_2 \mathbb{C}(x_1, \ldots, x_n)
\]
as right \( \mathbb{C}(x_1, \ldots, x_n) \)-modules. By [16, Corollary 0.5.5, p. 30] two polynomials \( h_1 \) and \( h_2 \) are stably associated if and only if there is a natural number \( d \geq 2 \) and \( P, Q \in \text{GL}_d(\mathbb{C}(x_1, \ldots, x_n)) \) such that
\[
h_1 \oplus I_{d-1} = P(h_2 \oplus I_{d-1})Q,
\]
where \( I_{d-1} \) is the identity matrix in \( M_{d-1}(\mathbb{C}) \).

Let \( g_1, \ldots, g_r \) be the atomic factors of \( g \) and let \( g_1 \) be the one that is associated to \( f \). By applying [28, Theorem 2.9, p. 8] we know that the hypersurfaces
\[
\text{Sing}_{k}(g_1), \ldots, \text{Sing}_{k}(g_r)
\]
are irreducible for \( k \in \mathbb{N} \) big enough, and we have
\[
\dim(\text{Sing}_{k}(g_i) \cap \text{Sing}_{k}(g_j)) < \dim(\text{Sing}_{k}(g_i)), \dim(\text{Sing}_{k}(g_j))
\]
by KRULL’s PRINCIPAL IDEAL THEOREM, i.e.
\[
\dim(\text{Sing}_{k}(f) \cap \text{Sing}_{k}(g_i)) = \dim(\text{Sing}_{k}(f))
\]
if and only if \( i = 1 \). Whence, if \( g_1 \) does not vanish on \( \text{Sing}(f) = \text{Sing}(g_1) \), then \( g \) cannot vanish on \( \text{Sing}(f) \). So we can assume that \( g = g_1 \).

If \( g \) vanishes on \( \text{Sing}(f) \), then so does \( \text{Le}(g) \). For sake of contradiction suppose that \( \text{Le}(g)(A_1, \ldots, A_n) \neq 0 \) for some \( (A_1, \ldots, A_n) \in \text{Sing}(f) \). Then \( \text{Le}(g)(tA_1, \ldots, tA_n) \in M_k(\mathbb{C}[t]) \) is not the zero matrix polynomial. Thus, for big \( t > 0 \) we will get \( g(tA_1, \ldots, tA_n) \neq 0 \). Note that \( (tA_1, \ldots, tA_n) \in \text{Sing}(f) \) for all \( t > 0 \).

Thus, we have \( \text{Sing}(g) \subseteq \text{Sing}(\text{Le}(g)) \), since \( \text{Le}(g) \) vanishes on \( \text{Sing}(f) \) and \( \text{Sing}(g) = \text{Sing}(f) \). The converse inclusion \( \text{Sing}(\text{Le}(g)) \subseteq \text{Sing}(g) \) must also hold: Suppose that there is a point \( (A_1, \ldots, A_n) \in M_k(\mathbb{C})^n \) such that \( (A_1, \ldots, A_n) \notin \text{Sing}(g) \), \( (A_1, \ldots, A_n) \in \text{Sing}(\text{Le}(g)) \) for \( k \in \mathbb{N} \) big enough. Then \( (tA_1, \ldots, tA_n) \in \text{Sing}(\text{Le}(g)) \) for all \( t > 0 \). The rank of \( g(tA_1, \ldots, tA_n) \) will equal the rank of \( \text{Le}(g)(tA_1, \ldots, tA_n) \) for big \( t > 0 \). On the

\[ \text{dim} (\text{Sing}_{k}(f) \cap \text{Sing}_{k}(g_i)) = \text{dim} (\text{Sing}_{k}(f)) \]

if and only if \( i = 1 \). Whence, if \( g_1 \) does not vanish on \( \text{Sing}(f) = \text{Sing}(g_1) \), then \( g \) cannot vanish on \( \text{Sing}(f) \). So we can assume that \( g = g_1 \).

If \( g \) vanishes on \( \text{Sing}(f) \), then so does \( \text{Le}(g) \). For sake of contradiction suppose that \( \text{Le}(g)(A_1, \ldots, A_n) \neq 0 \) for some \( (A_1, \ldots, A_n) \in \text{Sing}(f) \). Then \( \text{Le}(g)(tA_1, \ldots, tA_n) \in M_k(\mathbb{C}[t]) \) is not the zero matrix polynomial. Thus, for big \( t > 0 \) we will get \( g(tA_1, \ldots, tA_n) \neq 0 \). Note that \( (tA_1, \ldots, tA_n) \in \text{Sing}(f) \) for all \( t > 0 \).

Thus, we have \( \text{Sing}(g) \subseteq \text{Sing}(\text{Le}(g)) \), since \( \text{Le}(g) \) vanishes on \( \text{Sing}(f) \) and \( \text{Sing}(g) = \text{Sing}(f) \). The converse inclusion \( \text{Sing}(\text{Le}(g)) \subseteq \text{Sing}(g) \) must also hold: Suppose that there is a point \( (A_1, \ldots, A_n) \in M_k(\mathbb{C})^n \) such that \( (A_1, \ldots, A_n) \notin \text{Sing}(g) \), \( (A_1, \ldots, A_n) \in \text{Sing}(\text{Le}(g)) \) for \( k \in \mathbb{N} \) big enough. Then \( (tA_1, \ldots, tA_n) \in \text{Sing}(\text{Le}(g)) \) for all \( t > 0 \). The rank of \( g(tA_1, \ldots, tA_n) \) will equal the rank of \( \text{Le}(g)(tA_1, \ldots, tA_n) \) for big \( t > 0 \). On the
other hand side, \( \text{Le}(g)(tA_1, \ldots, tA_n) \) has not a full rank, while \( g(tA_1, \ldots, tA_n) \) has a full rank for big \( t > 0 \). Obviously, the both statements exclude each other, resulting in a contradiction.

So we reduced the situation to the point where we can assume that \( g = \text{Le}(g) \). Thus, \( \text{Le}(g) \) and \( f \) are stably associated. By Remark [27, Remark 4.1, p. 12] \( \text{Le}(g) \) must be of the form \( \lambda f \) for \( \lambda \in \mathbb{C}^* \), which results in a contradiction as \( \lambda f \) does not vanish on \( \text{Sing}(f) \) for \( \lambda \neq 0 \). Hence, \( V = \text{Sing}(f) \) is dense with respect to the hull-kernel topology and \( V_k \) is a lower dimensional non-empty variety for big \( k \in \mathbb{N} \).

Lemma 3.2.14. Let \( V \subseteq B(\mathcal{H})_{\text{her}}^n \) and \( \mathcal{H}_k \in \mathbb{G}_k(\mathcal{H}) \). For each \( k \) let \( P_k \) be the orthogonal projection onto \( \mathcal{H}_k \). Suppose that \( P_k^* V P_k \subseteq V \) for all \( k \in \mathbb{N} \). Then the following statements are equivalent:

a The set \( V \) is dense with respect to the hull-kernel topology.

b The operator \( f(A_1, \ldots, A_n) \in B(\mathcal{H}) \) vanishes for all \( (A_1, \ldots, A_n) \in V \) if and only if \( f = 0 \).

c The operator \( f(A_1, \ldots, A_n) \in B(\mathcal{H}) \) vanishes for all \( (A_1, \ldots, A_n) \in V \cap K(\mathcal{H})_{\text{her}}^n \) if and only if \( f = 0 \).

Proof: (a)⇒(b): Follows directly form Lemma [3.1.6]

(b)⇒(c): We must verify that the assertion of statement (b) is still true if we restrict ourselves to \( K(\mathcal{H})_{\text{her}} \). For the sake of contradiction suppose that we can find a polynomial \( f \in \mathbb{C}[x_1, \ldots, x_n]_{\text{her}} \setminus \{0\} \) such that

\[ f(A_1, \ldots, A_n) = 0 \]

for all \( (A_1, \ldots, A_n) \in V \cap K(\mathcal{H})_{\text{her}}^n \). By assumption there must be a point \( (B_1, \ldots, B_n) \in V \) with

\[ f(B_1, \ldots, B_n) \neq 0. \]

Consider

\[ \langle f(P_k^* B_1 P_k, \ldots, P_k^* B_n P_k)x, x \rangle = \langle f(B_1, \ldots, B_n) P_k x, P_k x \rangle \]

for \( x \in \mathcal{H} \). Since \( P_k^* A_1 P_k, \ldots, P_k^* A_n P_k \in B(\mathcal{H}_k) \), we get

\[ \langle f(P_k^* B_1 P_k, \ldots, P_k^* B_n P_k)x, x \rangle = \langle f(B_1, \ldots, B_n) P_k x, P_k x \rangle = 0. \]

As \( k \to \infty \) we get \( \langle f(B_1, \ldots, B_n) P_k x, P_k x \rangle \to \langle f(B_1, \ldots, B_n)x, x \rangle \). But \( \langle f(B_1, \ldots, B_n)x, x \rangle \neq 0 \) for at least one \( x \in \mathcal{H} \), resulting in a contradiction.

(c)⇒(b): That is obvious.

(b)⇒(a): Trivial.\[ \square \]
Theorem 3.2.15. (Operator version of stability) Let $z \in \mathbb{Z}^n$ and let $f_1, \ldots, f_r \in \mathbb{C}\langle x_1, \ldots, x_n \rangle_{\text{her}}$ be polynomials such that
\[
\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r) \neq 0
\]
are non-negative on a set $V$ with the following properties:

- The set $V$ is contained in $B(\mathcal{H})_{\text{her}}^n$ and for every orthogonal projection $P_k : \mathcal{H} \to \mathcal{H}$ into a subspace of dimension $k$ we have $P_k^* V P_k \subseteq V$.
- The set $V$ is dense with respect to the hull-kernel topology.

Then the quadratic module
\[
M^\text{nc} = QM\langle f_1, \ldots, f_r \rangle
\]
is totally stable with respect to the $z$-grading.

Proof: Without loss of generality, let $f \in M^\text{nc}$ be given by
\[
f = \sum_{ij} \text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}) + \text{Le}_z(s),
\]
where $s \in \mathbb{C}\langle x_1, \ldots, x_n \rangle^2$, $\text{Le}_z(q_{ij}) \neq 0$ for all $i, j$ and $\text{gr}_z(\text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij})) = \text{gr}_z(\text{Le}_z(s))$ for all $i, j$. Suppose that we have
\[
\sum_{ij} \text{Le}_z(q_{ij})^* (A_1, \ldots, A_n) \text{Le}_z(f_i)(A_1, \ldots, A_n) \text{Le}_z(q_{ij})(A_1, \ldots, A_n) + \text{Le}_z(s)(A_1, \ldots, A_n) = 0
\]
for all $(A_1, \ldots, A_n) \in V$. The denseness of $V$ and Lemma 3.2.14 imply that we have $\text{Le}_z(q_{ij})^* \text{Le}_z(f_i) \text{Le}_z(q_{ij}), \text{Le}_z(s) = 0$ for all $i, j$. Since $\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r) \neq 0$ we get $\text{Le}_z(q_{ij}) = 0$ for all $i, j$, which results in a contradiction. \qed

Remark 3.2.16. Suppose that we have the same prerequisites as in Theorem 3.2.15.

There is the following alternative proof of Theorem 3.2.15. Using the denseness of $V$ we argue that there is a point $(x, A_1, \ldots, A_n) \in \mathbb{C}^k \times V_k$ such that
\[
\langle \text{Le}_z(f_1)(A_1, \ldots, A_n)x, x \rangle > 0
\]
\[
\vdots
\]
\[
\langle \text{Le}_z(f_r)(A_1, \ldots, A_n)x, x \rangle > 0
\]
for $k \in \mathbb{N}$ big enough. Using that
\[
\langle \text{Le}_z(q)^* \text{Le}_z(f_i) \text{Le}_z(q)(A_1, \ldots, A_n)x, x \rangle = \langle \text{Le}_z(f_i)(A_1, \ldots, A_n) \text{Le}_z(q)(A_1, \ldots, A_n)x, \text{Le}_z(q)(A_1, \ldots, A_n)x \rangle \geq 0
\]
for any $q \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$ and $i = 1, \ldots, r$ it is not difficult to see that if $q \neq 0$ one can achieve at least one strict inequality. From there we immediately get that $M^\text{nc}$ has no degree cancellations.
The connection to the commutative case is the following: The inequalities \(3.2\) imply that the semialgebraic set \(S\) given by the quadratic forms

\[
\langle \text{Le}_z(f_1)(A_1, \ldots, A_n), \cdot \rangle, \ldots, \langle \text{Le}_z(f_r)(A_1, \ldots, A_n), \cdot \rangle
\]

has a non-empty interior. Inequality \(3.3\) tells us that \(S\) is invariant under transformation and by appropriately choosing \((A_1, \ldots, A_n) \in V_k\) we have that this point is also a point of \(S'((\text{Le}_z(f_i)(A_1, \ldots, A_n)), \cdot)\) for some \(i = 1, \ldots, r\).

**Theorem 3.2.17. (Matrix version of stability)** Let \(z \in \mathbb{Z}^n\) and let \(f_1, \ldots, f_r \in \mathbb{C} \langle x_1, \ldots, x_n \rangle_{\text{her}}\) be polynomials such that

\[
\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)
\]

are non-negative on a set \(V \subseteq K(H)_{\text{her}}^n\) which is dense with respect to the hull-kernel topology and satisfies \(V = \bigcup_{k=1}^{\infty} V_k\). Then the quadratic module

\[
\text{QM}(f_1, \ldots, f_r)
\]

is totally stable with respect to the \(z\)-grading.

**Proof:** Same as Theorem 3.2.15 but without the need of using Lemma 3.2.14 \(\square\)

### 3.3 Non-commutative semi-algebraic sets and tentacles

Till now we analyzed the properties of a finitely generated quadratic module \(M^\infty\) by using the \(z\)-grading and the hull-kernel topology. This lead finally to Theorem 3.2.15. Since we evaluate non-commutative polynomials on more complicated spaces as \(\mathbb{C}^n\) or \(\mathbb{R}^n\), it is clear that just looking at the \(z\)-grading is inadequate. Indeed, increasing the size of matrices increases the dimension of the vector space where we evaluate the polynomials. For example, let \(S^\infty_{k}\) be a non-commutative semialgebraic set. For big \(k\) the set \(S^\infty_{k}\) might have far more directions where it goes to infinity than \(S^\infty_{1}\). This motivates to define a notion of a tentacle in the non-commutative case, which we will do in the following.

**Remark 3.3.1.** There are also non-commutative semialgebraic \(S^\infty\) sets, those number of directions at which \(S^\infty\) tends to infinity is independent of the level \(k\). Let us consider the following simple case: Let \(f = 1-x_1^2-x_2^2\). We interested in all matrices \(A_1, A_2 \in \text{Her}_n\) such that \(f(A_1, A_2) \geq 0\).

It is easy to see that the set \(\{(A_1, A_2) \in \text{Her}_n^2 : f(A_1, A_2) \geq 0\}\) is contained in \(\{(A_1, A_2) \in \text{Her}_n^2 : 0 \leq \|A_1\|, \|A_2\| \leq 1\}\). More generally we get:

**Lemma 3.3.2.** For \(f = 1-x_1^2-\cdots-x_n^2\) and for every \(k \in \mathbb{N}\), we have that \(S_k\) is contained in the unitbox of \(M_k(\mathbb{C})^n\). Furthermore, \(S\) is a bounded set in \(K(H)^n\).
Remark 3.3.3. Lemma 3.3.2 remains true for a certain class of semialgebraic sets that are described by [25, Theorem 1.4, p. 5].

Let us now introduce the notion of TENTACLES FOR SUBSETS OF $K(\mathcal{H})^n$. Consider the subset $V \subseteq K(\mathcal{H})^n$. The set $MT(V)$ of tentacles of $V$ is given by the following data:

- A family of sets $(MT_k(V_k))_k$.
- Each set $MT_k(V_k)$ consists of sets $MT_k$ that are given by compact subsets $B_k$ of $H_k^n$ (which we call a BASIS of $MT_k$) and $\Phi_1, \ldots, \Phi_n \in \text{Sym}_k \otimes \mathbb{R}(t)$ such that
  
  $$MT_k := \{ (\Phi_1(t) \otimes X_1, \ldots, \Phi_n(t) \otimes X_n) : (X_1, \ldots, X_n) \in B_k, t \geq 1, \Phi_1(t), \ldots, \Phi_n(t) \text{ exist} \}$$

  is contained in $V_k$, where $\otimes$ stands for component-wise multiplication. We call the functions $\Phi_1, \ldots, \Phi_n$ the DEFINING FUNCTIONS OF $MT_k$. If no such set can be found, then we set $MT_k(V_k) = \emptyset$.

By using the identifications

$$\text{Sym}_k \otimes \mathbb{R}(t) \cong \mathbb{R}^{\frac{k(k+1)}{2}} \otimes \mathbb{R}(t) \cong \mathbb{R}(t)^{\frac{k(k+1)}{2}},$$

we can transport all definitions we made with respect to the ordinary tentacles to the matrix-tentacles.

Remark 3.3.4. Let $V \subseteq K(\mathcal{H})^n$ and let $MT \in MT(V)$.

- A fibre $F$ of $MT$ is a sequence of sets
  
  $$\text{Fibre}_{B,k}(MT_k) := \{ (\Phi_1(t) \otimes B_1, \ldots, \Phi_n(t) \otimes B_n) : t \geq 1, \Phi_1(t), \ldots, \Phi_n(t) \text{ are defined} \},$$

  where $B \in B_k$. We will shorten the notation by just writing $(F_k)_k$ for the sequence.

- For each $k \in \mathbb{N}$ we define $\text{deg}(MT_k)$ to be the tuple in $(\mathbb{Z}^{k \times k})^n$ that we get by applying the degree mapping entry wise onto the tuple of defining functions of $MT_k$.

- We define $MT^+(V)$ to be the family of sets $(MT^+_k(V_k))_k$, where $MT^+_k(V_k)$ is the subset of $MT_k(V_k)$ that consists of all $MT_k \in MT_k(V_k)$ for which at least entry of $\text{deg}(MT_k)$ is positive.

- Now we can define $c(V)$: For each $k \in \mathbb{N}$ the number $c(V_k)$ is defined to be the biggest natural number $l$ such that we can find $MT_{1,k}, \ldots, MT_{l,k} \in MT_k(V_k)$ with
  
  $$\mathbb{Z} \text{deg}(MT_{1,k}) + \cdots + \mathbb{Z} \text{deg}(MT_{l,k}) = \mathbb{Z} \text{deg}(MT_{1,k}) \oplus \cdots \oplus \mathbb{Z} \text{deg}(MT_{l,k}) \cong \mathbb{Z}^l.$$

  Finally, we set $c(V) = \lim_{k \to \infty} c(V_k)$.

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Definition 3.3.5. The map that sends an element \( z = (z_1, \ldots, z_n) \in \mathbb{Z}^n \) to
\[
\begin{pmatrix}
  z_1 & \cdots & z_1 \\
  \vdots & \ddots & \vdots \\
  z_1 & \cdots & z_1
\end{pmatrix}, \ldots,
\begin{pmatrix}
  z_n & \cdots & z_n \\
  \vdots & \ddots & \vdots \\
  z_n & \cdots & z_n
\end{pmatrix}
\in (\mathbb{Z}^{k \times k})^n
\]
is called the diagonal map.

In the commutative case the tentacle was somewhat the geometric interpretation of a \( z \)-grading. This correspondence is not true with respect to the matrix tentacles. For example, let \( z \in \mathbb{N}^2 \) and consider a \( z = (z_1, z_2) \)-grading of \( \mathbb{C}(x_1, x_2) \). The vector space isomorphism \( \mathbb{C} x_1 \sim \mathbb{C} x_1, x_2, x_1 \mapsto x_1 x_2 \) is of degree \( z_2 \) as it increases the degree of the preimage by \( z_2 \). Take \( MT \in MT(K(\mathcal{H})_\text{herm}) \). Fix a number \( k \in \mathbb{N} \) and let \( MT_k \) be given by \( \Phi_1, \Phi_2 \in \mathbb{R}(t)^{k \times k} \) of degree
\[
\deg(MT_k) = (D_1, D_2) \in \mathbb{Z}^{k \times k} \times \mathbb{Z}^{k \times k}
\]
such that \( z_1 \) appears in \( D_1 \in \mathbb{Z}^{k \times k} \) and \( z_2 \) appears in \( D_2 \in \mathbb{Z}^{k \times k} \). Furthermore, we demand that \( z_1 \) appears in \( D_1 \) and \( z_2 \) appears in \( D_2 \), where both \( z_1 \) and \( z_2 \) are the biggest entries in \( D_1 \) resp. \( D_2 \). To simplify the situation further we will assume that \( D_1 \) and \( D_2 \) have only non-zero entries on their diagonals. Even under the assumptions above, the vector space isomorphism
\[
\mathbb{C} \Phi_1 \sim \mathbb{C} \Phi_2, \Phi_1 \mapsto \Phi_1 \Phi_2
\]
is not compatible with the degree increase of the \( z \)-grading above, since it is possible that \( z_1 + z_2 \) is strictly greater than any entry of \( \deg(\Phi_1 \Phi_2) \), as one can easily verify. This is the reason why the degree-grading correspondence does not work anymore. Dealing with this incompatibility will be our major concern in this section.

Proposition 3.3.6. Let \( S = S(f_1, \ldots, f_r) \subseteq \mathbb{R}^n \) be a semialgebraic set such that \( z \in \deg(T(S)) \). Consider the associated non-commutative semialgebraic set \( S^{nc} = S^{nc}(f_1, \ldots, f_r) \). Then there is an element \( MT \in MT(S^{nc}) \) such that \( z = \deg(MT_k) \) for all \( k \in \mathbb{N} \), where \( z \) is embedded into \( \mathbb{Z}^{(n+1)[n+1]} \) via
\[
\mathbb{Z}^n \mapsto \mathbb{Z}^{n(k \times k)}(z_1, \ldots, z_n) \mapsto \begin{pmatrix}
  z_1 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & z_n
\end{pmatrix}
\]

\[
\begin{pmatrix}
  z_n & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & z_n
\end{pmatrix}
\]

Proof: Let \( T \in T(S) \) together with its defining functions \( (\varphi_1, \ldots, \varphi_n) \) and basis \( B \subseteq S \).

Without loss of generality, we can assume that for each \( (x_1, \ldots, x_n) \in B \) we always have \( x_1, \ldots, x_n \neq 0 \). Let \( z = \deg(T) \). We lift \( T \) to an element \( MT \) in \( MT(S^{nc}) \) by sending each \( \varphi_i \) to \( \Phi_i = \begin{pmatrix}
  \varphi_i & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & \varphi_i
\end{pmatrix} \) for each \( i = 1, \ldots, n \) and each \( k \in \mathbb{N} \) and by repeating the
same procedure for \( B \subseteq \mathbb{R}^n \). To be more precise, the set \( B \) gives rise to a basis \( B_k \) of \( MT_k \) because the defining functions only have non-zero entries on the diagonals and therefore one can easily extend the embedding of \( B \) such that it has a non-empty interior. Then it is easy to see that for every \((A_1, \ldots, A_n) \in B_k\) that represents \((x_1, \ldots, x_n) \in B\) we get

\[
f_i(\Phi_1 \odot A_1, \ldots, \Phi_n \odot A_n) = \begin{pmatrix}
f_i(\varphi_1 x_1, \ldots, \varphi_n x_n) & \cdots & 0 \\
\vdots & \ddots & \vdots \\
0 & \cdots & f_i(\varphi_1 x_1, \ldots, \varphi_n x_n)
\end{pmatrix}
\]

for every \( i = 1, \ldots, r \). The above equation implies the claim.

The question now is, if \( MT_2(S^{nc}) \) can be identified with \( T^+(S) \) or not. To answer this question consider the polynomial \( f = -x_1^2 x_2^2 + x_1 x_2 \in \mathbb{R}[x_1, x_2] \). The corresponding semi-algebraic set \( S := S(f) \subseteq \mathbb{R}^2 \) is not compact.

\[
\begin{align*}
\begin{array}{|c|c|}
\hline
x_1 & x_2 \\
\hline
\end{array}
\end{align*}
\]

Let us consider now \( S^{nc} \), where \( S^{nc} = S(f) \). Set \( \Phi_1(t) = \begin{pmatrix} t^{c_{11}} & 0 \\ 0 & t^{c_{22}} \end{pmatrix} \) and \( \Phi_2(t) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \). Let \( A_1 = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \) resp. \( A_2 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \) be such that \((A_1, A_2) \in \text{int}(S^{nc})\) and \( 0 < a_{11}, a_{22}, b_{11}, b_{12}, b_{22} < 1 \). Then

\[
f(\Phi_1 \odot A_1, \Phi_2 \odot A_2) = \begin{pmatrix}
a_{11} b_{11} t^{c_{11} + d_{11}} - b_{11}^2 t^{c_{11} + d_{11}} \\
\frac{1}{2} (a_{12} b_{11} t^{c_{11} + d_{12}} + a_{12} b_{22} t^{c_{22} + d_{12}}) \\
a_{22} b_{22} t^{c_{22} + d_{22}} - b_{22}^2 a_{22} t^{c_{22} + d_{22}}
\end{pmatrix}
\]

Furthermore, we demand that

\[
c_{11} > -d_{11}, d_{12} < d_{22} < c_{11} < c_{22} < 0.
\]

Then \( a_{11} b_{11} a_{22} b_{22} t^{c_{22} + d_{22} + c_{11} + d_{11}} \) is the leading term of \( \det(f(\Phi_1(t) \odot A, \Phi_2(t) \odot B)) \).

For \( t \geq 1 \) big enough we get

\[
a_{11} b_{11} t^{c_{11} + d_{11}} - b_{11}^2 a_{11} t^{c_{11} + d_{11}} > 0,
\]

\[
a_{22} b_{22} t^{c_{22} + d_{22}} - b_{22}^2 a_{22} t^{c_{22} + d_{22}} > 0
\]

and \( \det(f(\Phi_1(t) \odot A, \Phi_2(t) \odot B)) > 0 \) resp. \( f(\Phi_1(t) \odot A, \Phi_2(t) \odot B) > 0 \).
Whence \( c(S^{nc}_{2}) > c(S) \), which implies not just that \( \text{MT}^{+}_{2}(S^{nc}_{2}) \) and \( T^{+}(S) \) cannot be identified, but also that this holds true if we fix the bases. Finally, we will consider \( c(S^{nc}) \): We start with \( \Phi_{1}^{(1)}, \Phi_{2}^{(1)}, A_{1}^{(1)} \) and \( A_{2}^{(1)} \) as above. Then for the next step, we demand that either \( \Phi_{1}^{(2)} = \Phi_{1}^{(1)} \) resp. \( \Phi_{2}^{(2)} = \Phi_{2}^{(1)} \) or that \( \Phi_{1}^{(2)} = \Phi_{2}^{(2)} = 0 \). We apply the same construction on the matrices \( A_{1}^{(2)} \) resp. \( A_{2}^{(2)} \). Inductively we get

\[
\left( \bigoplus_{i=1}^{k} \left( \Phi_{1}^{(i)} \cdot A_{1}^{(i)} \right), \bigoplus_{i=1}^{k} \left( \Phi_{2}^{(i)} \cdot A_{2}^{(i)} \right) \right) \in \mathbb{R}(t)^{2k \times 2k} \times \mathbb{R}(t)^{2k \times 2k}.
\]

By what we know, the above construction gives rise to an element in \( \text{MT}^{+}_{k}(S^{nc}_{2k}) \). Varying through the different possibilities of the matrices \( \Phi_{1}^{(i)} \) and \( \Phi_{2}^{(i)} \) leads to the conclusion that \( c(S^{nc}_{2k}) \) must be at least \( k \). But that implies \( c(S^{nc}) = \infty \).

**Remark 3.3.7.** For a non-commutative semialgebraic set \( S^{nc} \) we defined \( c(S^{nc}) \) to be the \( \lim_{k \to \infty}(S^{nc}_{k}) \). The motivation behind it is as follows: Consider the sequence

\[
\mathbb{Z}c(S^{nc}_{1}), \ldots, \mathbb{Z}c(S^{nc}_{k}), \ldots
\]

and the resulting \( \mathbb{Z} \)-module \( \mathbb{Z}c(S^{nc}) \). In general, this module does not need to be free, in fact, it does not even need to be projective. For example consider the case, where \( S^{nc} \) is the whole space and therefore \( \mathbb{Z}c(S^{nc}) \cong \mathbb{Z}^{\mathbb{N}} \) which is not a free \( \mathbb{Z} \)-module resp. it is not even projective. Hence, it is not possible to speak of independent directions in this case.

Or, in other words, Definition 1.2.1 cannot be modified to work for \( S^{nc} \). One way would be to replace \( \mathbb{Z} \) with \( \mathbb{R} \) in Definition 1.2.1 But it is far more complicated to determine the \( \mathbb{R} \)-vector space that would correspond to \( c(S^{nc}) \). To summarize up, this shows how much more complicated the geometry of \( S^{nc} \) is than the one of \( S^{nc}_{1} \).

**Theorem 3.3.8.** (Exclusion I) Let \( S = S(f_{1}, \ldots, f_{r}) \subseteq \mathbb{R}^{2} \) be a semialgebraic, \( S^{nc} \) the corresponding set in \( K(\mathcal{H})_{\text{her}}^{2} \) and \( V = \bigcup_{k} \text{Her}^{n}_{\text{her}}^{k} \). Fix some \( z \in \mathbb{Z}^{2} \) and \( k \in \mathbb{N} \) and write

\[
\text{Le}_{z}(f_{i}) = \sum_{\alpha_{i}} a_{\alpha_{i}} x_{1}^{\alpha_{1}} x_{2}^{\alpha_{2}},
\]

for \( i = 1, \ldots, r \). Suppose that the following conditions are satisfied:

a) The intersection

\[
\text{SRow}_{\alpha_{i}}(f_{1}, \ldots, f_{r}) \cap \left\{ (x_{1}, \ldots, x_{2k}) \in \mathbb{R}^{2k} : x_{1}, \ldots, x_{2k} \geq 0 \right\}
\]

is almost bounded.

b) Let

\[
((x_{1}, y_{1}), \ldots, (x_{k}, y_{k})), ((x_{1}, y_{1}), \ldots, (x_{k}, y_{1})) \in \left\{ (x_{1}, \ldots, x_{2k}) \in \mathbb{R}^{2k} : x_{1}, \ldots, x_{2k} \geq 0 \right\}
\]

for \( i = 1, \ldots, r \). Suppose that the following conditions are satisfied:

a) The intersection

\[
\text{SRow}_{\alpha_{i}}(f_{1}, \ldots, f_{r}) \cap \left\{ (x_{1}, \ldots, x_{2k}) \in \mathbb{R}^{2k} : x_{1}, \ldots, x_{2k} \geq 0 \right\}
\]

is almost bounded.
such that $0 < x_1 \leq \cdots \leq x_k$ and $0 < y_1 \leq \cdots \leq y_k$. If

$$\sum_{j=1}^{k} \text{Le}_2(f_i)(x_j, y_j), \sum_{j=1}^{k} \text{Le}_2(f_i)(x_j, y_{k-j+1}) < 0$$

for some $i = 1, \ldots, r$, then for all $0 \leq t_{\alpha, i} \leq 1$ we have

$$\sum_{j=1}^{k} \sum_{\alpha_i} a_{\alpha_i} x_j^{\alpha_1} \gamma_j(t_{\alpha, i})^{\alpha_2} < 0,$$

where $\gamma_j : [0,1] \to \mathbb{R}, t \mapsto ty_j + (1-t)y_{k-j+1}$.

Let $MT \in MT^+(V)$. If $\deg(MT_k) = z$, where $z$ is taken under the diagonal mapping, then $MT_k$ is not contained in $MT_k^+(S^{nc} k)$.

**Proof:** Step I: Let $MT_k \in MT_k(V_k)$ be given by $\Phi_1, \Phi_2 \in (\text{Sym}_k \otimes \mathbb{R}(t))^2$ with

$$\deg(\Phi_1) = \begin{pmatrix} z_{11} & \cdots & z_{1k} \\ \vdots & \ddots & \vdots \\ z_{k1} & \cdots & z_{kk} \end{pmatrix}, \deg(\Phi_2) = \begin{pmatrix} z'_{11} & \cdots & z'_{1k} \\ \vdots & \ddots & \vdots \\ z'_{k1} & \cdots & z'_{kk} \end{pmatrix}$$

and

$$(\deg(\Phi_1), \deg(\Phi_2)) = z.$$

Since the degree is taken to be $z$ under the diagonal map, we get $z_{11} = \cdots = z_{kk}$ and $z'_{11} = \cdots = z'_{kk}$. This condition has the following pleasant implication: Since we will only be interested in big values, we can assume that

$$\Phi_1 = \begin{pmatrix} t^{z_{11}} & \cdots & t^{z_{11}} \\ \vdots & \ddots & \vdots \\ t^{z_{11}} & \cdots & t^{z_{11}} \end{pmatrix}, \Phi_2 = \begin{pmatrix} t^{z_{22}} & \cdots & t^{z_{22}} \\ \vdots & \ddots & \vdots \\ t^{z_{22}} & \cdots & t^{z_{22}} \end{pmatrix}$$

and that $f_1 = \text{Le}_2(f_1), \ldots, f_r = \text{Le}_2(f_r)$.

Step II: Let $(A_1, A_2) \in S^{nc} k$. The eigenvalues of $\Phi_1 \otimes A_1$ resp. $\Phi_2 \otimes A_2$ are continuous functions

$$\lambda_1(t), \ldots, \lambda_k(t), \lambda'_1(t), \ldots, \lambda'_k(t)$$

such that there is a value $t' \geq 1$ with $\lambda_1(t) \leq \cdots \leq \lambda_k(t)$ resp. $\lambda'_1(t) \leq \cdots \leq \lambda'_k(t)$ for all $t \geq t'$. In fact, the functions $\lambda_1(t), \ldots, \lambda_k(t), \lambda'_1(t), \ldots, \lambda'_k(t)$ are all rationals. The rationals $\lambda_1(t), \ldots, \lambda_k(t)$ are of degree $z_1$ and the rationals $\lambda'_1(t), \ldots, \lambda'_k(t)$ are of degree $z_2$.

Step III: We will now verify that $MT_k \not\in MT_k^+(S^{nc} k)$. Since $\text{Her}_k \cap \text{GL}_k(\mathbb{C})$ is dense in $\text{Her}_k$, we can assume that $(\Phi_1(t) \otimes A_1, \Phi_2(t) \otimes A_2) \in \text{GL}_k(\mathbb{C})^2$ for $t \geq t'$. Consider

$$\text{tr}(f_i(\Phi_1(t) \otimes A_1, \Phi_2(t) \otimes A_2)) = \sum_{\alpha} a_{\alpha} \text{tr} ((\Phi_1(t) \otimes A_1)^{\alpha_1} (\Phi_2(t) \otimes A_2)^{\alpha_2})).$$
By [33], Theorem II.2, p. 1501 we have
\[
\sum_{j} \lambda_j(t)^{\alpha_1} \chi_{k-j+1}(t) t^{\alpha_2} \leq \text{tr} \left( \left( (\Phi_1(t) \odot A_1)^{\alpha_1} (\Phi_2(t) \odot A_2)^{\alpha_2} \right) \right) \leq \sum_{j} \lambda_j(t)^{\alpha_1} \chi_j(t) t^{\alpha_2}.
\] (3.4)

We can assume that each \(\lambda_i(t)\) resp. \(\chi_i(t)\) can be written as
\[
\lambda_i(t) = b_i t^{x_1}, \chi_i(t) = b'_i t^{x_2}
\]
for \(i = 1, \ldots, k\). Furthermore, we have
\[
\sum_{\alpha_i} \sum_{j} \alpha_i \lambda_j(t)^{\alpha_1} \chi_{k-j+1}(t) t^{\alpha_2} = \sum_{j} f_i(\lambda_j(t), \chi_{k-j+1}(t))
\]
resp.
\[
\sum_{\alpha_i} \sum_{j} \lambda_j(t)^{\alpha_1} \chi_j(t) t^{\alpha_2} = \sum_{j} f_i(\lambda_j(t), \chi_j(t)).
\]
Condition (a) grantees that both sums become negative for big \(t \geq 1\), some appropriate \((A_1, A_2) \in S_{\text{nc}}^k\) and at least one \(i = 1, \ldots, r\). Inequality 3.4 and condition (b) now imply that
\[
\sum_{\alpha_i} \text{tr} \left( \left( (\Phi_1(t) \odot A_1)^{\alpha_1} (\Phi_2(t) \odot A_2)^{\alpha_2} \right) \right) < 0
\]
for big \(t \geq 1\) and at least one \(i = 1, \ldots, r\). Thus, \(MT_k\) is not contained in \(S_{\text{nc}}^k\) finishing the proof. \(\square\)

Remark 3.3.9. The prerequisites of Theorem 3.3.8 are quite strong and it is difficult to loosen them. Suppose we extend our statement to matrix tentacles \(MT \in MT(V) \cup MT(-V)\), with \(V = \bigcup_{n \in \mathbb{N}} \text{Her}^+_n\). The first problem we get is that inequality 3.4 does not give us information about the sign of \(\text{tr} \left( \left( (\Phi_1(t) \odot A_1)^{\alpha_1} (\Phi_2(t) \odot A_2)^{\alpha_2} \right) \right)\) anymore. That is because the left-hand side of the inequality can be negative while the right side of the inequality can be positive. So the proof of Theorem 3.3.8 does not work anymore because we lose the ability to control the trace.

Thus, there is not much room for improvement if we approach this problem by using the trace. Alternatively we could consider \(\text{tr} \left( \left( (\Phi_1(t) \odot A_1)^{\alpha_1} (\Phi_2(t) \odot A_2)^{\alpha_2} \right) \right)\) without the trace. However, we must then deal with eigenvalues of products and sums of hermitian matrices, which is even more uncontrollable.

Example 3.3.10. Let \(f = 7 - x_1 - x_2\) and \(k = 2\). Then \(\text{SRow}_{1,2}(f) \cap \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4, x_1, x_2, x_3, x_4 \geq 0\}\) is bounded. Thus condition (a) of Theorem 3.3.8 is satisfied. Condition (b) is verified as follows: Consider \(f' = f(x_1, x_2) + f(x_3, x_4) = 14 - x_1 - x_2 - x_3 - x_4\). Fix a point \((x_1, x_2) \in \mathbb{R}^2\) with \(x_1, x_2 \geq 0\) and let \(x_2 \leq x_4\) with \(x_2, x_4 \geq 0\) and \(f'(x_1, x_2, x_3, x_4), f'(x_1, x_4, x_3, x_2) < 0\). Let \(L \subseteq \mathbb{R}\) be the line that connects \(x_2\) and \(x_4\). Then
\[
\text{SRow}_{1,2}(f) \cap \{(x_1) \times L \times \{x_3\} \times L\} = \varnothing,
\]
implying that condition (b) is also satisfied.
Proposition 3.3.11. Let \( A \in M_n(\mathbb{C}) \). Then the following statements are equivalent:

(i) \( A \) is positive.

(ii) The map \( M_n(\mathbb{C}) \to M_n(\mathbb{C}), B \mapsto A \odot B \) is positive.

(iii) The map \( M_n(\mathbb{C}) \to M_n(\mathbb{C}), B \mapsto A \odot B \) is completely positive.

Proof: See [43, Theorem 3.7, p. 31].

Let us consider the sequence

\[
(\text{Sym}_k \otimes \mathbb{R}(t) ) \times \text{Her}_k \to \text{Her}_k \otimes \mathbb{R}(t) \to M_k(\mathbb{Z}),
\]

where the first map \((\text{Sym}_k \otimes \mathbb{R}(t) ) \times \text{Her}_k \to \text{Her}_k \otimes \mathbb{R}(t)\) is given by the componentwise multiplication and the second map \(\text{Her}_k \otimes \mathbb{R}(t) \to M_k(\mathbb{Z})\) is given by the degree mapping. By using Proposition [3.3.11] we have the following property of the first map:

Let \((\text{Sym}_k \otimes \mathbb{R}(t) ) \times \text{Her}_k \to \text{Her}_k \otimes \mathbb{R}(t)\) be the valuation maps for some arbitrary big \( t \in \mathbb{R} \). The commutative diagram

\[
\begin{array}{c}
(\text{Sym}_k \otimes \mathbb{R}(t) ) \times \text{Her}_k \\
\downarrow \quad \downarrow \text{Her}_k \\
\text{Her}_k \otimes \mathbb{R}(t)
\end{array}
\]

will evaluate in the psd cone of \( \text{Her}_k \) for big \( t \in \mathbb{R} \), if the tuple in \((\text{Sym}_k \otimes \mathbb{R}(t) ) \times \text{Her}_k \) is positive for big \( t \in \mathbb{R} \). With other words, these maps are preserving positivity. It remains to investigate \( \text{Her}_k \otimes \mathbb{R}(t) \to M_k(\mathbb{Z}) \). One interesting thing is to figure out how this maps behaves under multiplication, i.e what is \( \deg(\Phi_1 \Phi_2) \), where \( \Phi_1, \Phi_2 \in \text{Her}_k \otimes \mathbb{R}(t) \). Let \( D_1 = \deg(\Phi_1) = (a_{ij}) \) and \( D_2 = \deg(\Phi_2) = (b_{ij}) \). From these two matrices \( D_1 \) and \( D_2 \) we create

\[
C(\Phi_1, \Phi_2) = \begin{pmatrix}
\max\{a_{1i} + b_{1i} : i\} & \cdots & \max\{a_{ni} + b_{ni} : i\} \\
\vdots & \ddots & \vdots \\
\max\{a_{ni} + b_{1i} : i\} & \cdots & \max\{a_{ni} + b_{ni} : i\}
\end{pmatrix}.
\]

We are ready to prove some basic properties of these matrices:

Lemma 3.3.12. Let \( \Phi_1, \Phi_2 \in \text{Her}_k \otimes \mathbb{R}(t) \) and let \( \Phi_1 \in U_1 \) and \( \Phi_2 \in U_2 \) be some arbitrary small neighborhoods in the \( \mathbb{R} \)-vector space \( \text{Her}_k \otimes \mathbb{R}(t) \). Then we have the following statements:

a) There are \( \Phi_1' \in U_1 \) and \( \Phi_2' \in U_2 \) such that \( C(\Phi_1', \Phi_2') = \deg(\Phi_1' \Phi_2') \) holds. Furthermore, each entry of \( C(\Phi_1, \Phi_2) \) is bigger or equal as the corresponding entry in \( \deg(\Phi_1 \Phi_2) \).
b Let $\Phi \in \text{Sym}_k \otimes \mathbb{R}(t)$. If there is a number $t \geq 0$ such that $\Phi(t') \geq 0$ for all $t' \geq t$, then the biggest (not necessary unique) entry of $\deg(\Phi)$ is on the diagonal.

Proof: (a): That is clear.

(b): Let $g \in \mathbb{R}[t]$ be a polynomial such that $g\Phi \in \mathbb{R}[t]^{k \times k}$. If the biggest entry of $\deg(g\Phi)$ is on the diagonal, then the biggest entry of $\deg(\Phi)$ is also on the diagonal. The reverse is also true. Thus, we can assume that $\Phi \in \mathbb{R}[t]^{k \times k}$. By taking the leading terms we may assume that $\Phi$ is of the form

$$
\Phi = \begin{pmatrix}
  a_{11}t^{z_{11}} & \cdots & a_{1k}t^{z_{1k}} \\
  \vdots & \ddots & \vdots \\
  a_{k1}t^{z_{k1}} & \cdots & a_{kk}t^{z_{kk}}
\end{pmatrix}
$$

and accordingly to the above equation we get

$$
\deg(\Phi) = \begin{pmatrix}
  z_{11} & \cdots & z_{1k} \\
  \vdots & \ddots & \vdots \\
  z_{k1} & \cdots & z_{kk}
\end{pmatrix}.
$$

For large $t \geq 1$ the matrix $\Phi(t)$ is positive definite and therefore

$$
\det \begin{pmatrix}
  a_{ii}t^{z_{ii}} & a_{ij}t^{z_{ij}} \\
  a_{ij}t^{z_{ij}} & a_{jj}t^{z_{jj}}
\end{pmatrix} = a_{ii}a_{jj}t^{z_{ii} + z_{jj}} - a_{ij}^2t^{2z_{ij}} \geq 0
$$

for all $i, j \in \{1, \ldots, k\}$. But that implies immediately that the biggest entry of $\deg(\Phi)$ is on the diagonal. \qed

Definition 3.3.13. A tuple $(\Phi_1, \ldots, \Phi_n) \in (\text{Sym}_k \otimes \mathbb{R}(t))^n$ is called compatible with respect to $\max$ if

$$
\max(\deg(\Phi_1), \ldots, \Phi_n)) = \max(\deg(\Phi_1)) + \cdots + \max(\deg(\Phi_n))
$$

Remark 3.3.14. Here are some obvious facts about compatible tuples:

- Every tuple $(\Phi_1, \ldots, \Phi_n) \in (\text{Sym}_k \otimes \mathbb{R}(t))^n$ with $\Phi_1(t), \ldots, \Phi_n(t) \geq 0$ for big $t$ and $\Phi_1 = \cdots = \Phi_n$ is compatible.

- If $(\Phi_1, \ldots, \Phi_n)$ is compatible with $\Phi_1(t), \ldots, \Phi_n(t) \geq 0$ for big $t$, then $(\Phi_1^{\alpha_1}, \ldots, \Phi_n^{\alpha_n})$ is also compatible for all $\alpha_1, \ldots, \alpha_n \in \mathbb{N}$.

- Let $(\Phi_1, \ldots, \Phi_n)$ be compatible and let $z = (\max(\deg(\Phi_1)), \ldots, \max(\deg(\Phi_n))) \in \mathbb{Z}^n$. Then $z$ gives rise to a grading of $\mathbb{C}(x_1, \ldots, x_n)$. Let $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \mathbb{C}(x_1, \ldots, x_n)$ and consider the sequence
of vector spaces. Each one of the column isomorphisms
\[ C^{a_1}_x \xrightarrow{\sim} C^{a_1}_x \phi_2 \xrightarrow{\sim} \cdots \xrightarrow{\sim} C^{a_1}_x \phi_n \]
gives rise to one of the commutative squares
\[ \begin{array}{ccc}
C^{a_1}_x & \xrightarrow{\phi} & C \phi_1^{a_1} \phi_2^{a_2} \\
adeg & \downarrow & (deg)_{\max} \\
\mathbb{Z} & \xrightarrow{id} & \mathbb{Z}
\end{array} \]

In other words, the degree shifts of the sequence
\[ C^{a_1}_x \xrightarrow{\sim} \cdots \xrightarrow{\sim} C^{a_1}_x \phi_n \]
with respect to the \( z \)-degree are the same as the degree shifts of sequence
\[ C \phi_1^{a_1} \xrightarrow{\sim} \cdots \xrightarrow{\sim} C \phi_1^{a_1} \phi_n \]
with respect to the composition \( \max(deg) \).

- Let \( f \in \mathbb{C}(x_1, \ldots, x_n) \) be the canonical lift of a commutative polynomial in \( \mathbb{C}[x_1, \ldots, x_n] \).
  We say that \( f \) is compatible if
  \[ (\Phi_1^{a_1}, \ldots, \Phi_n^{a_n}) \]
is a compatible tuple for all \( \Phi_1, \ldots, \Phi_n \in (\text{Sym}_k \otimes \mathbb{R}(t))^n \) with \( \Phi_1(t), \ldots, \Phi_n(t) \geq 0 \)
for \( t \geq 1 \) big enough, and all \( (\alpha_1, \ldots, \alpha_n) \) such that \( x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in \text{Monomial}(f) \).

- For any tuple \( (\Phi_1, \ldots, \Phi_n) \in (\text{Sym}_k \otimes \mathbb{R}(t))^n \) we have the inequality
  \[ \max(deg(\Phi_1) + \cdots + deg(\Phi_n)) \leq \max(deg(\Phi_1)) + \cdots + max(deg(\Phi_n)). \]

**Lemma 3.3.15.** Let \( f \in \mathbb{C}(x_1, \ldots, x_n)_{\text{ker}} \) be a polynomial that arises from the canonical lift a commutative polynomial. The following statements are equivalent:

a The polynomial \( f \) is compatible.

b The polynomial \( f \) is in the image of the mapping
\[ \mathbb{C}(x_1, \ldots, x_n) \to \mathbb{C}[x_1, \ldots, x_n], g \mapsto g(x_1, 0, \ldots, 0) + \cdots + g(0, \ldots, 0, x_n) \]
Proof: (a)⇒(b): Suppose that we can find a monomial \( m \in \text{Monomial}(f) \) such that \( x_i^{a_i} x_j^{a_j} \) or \( x_j^{a_j} x_i^{a_i} \) appears in it. The ring \( \text{Sym}_k \otimes \mathbb{R}(t) \) has a lot zero divisors and therefore it is easy to find \( (\Phi_1, \ldots, \Phi_n) \in (\text{Sym}_k \otimes \mathbb{R}(t))^n \) such \( \Phi_i^{a_i} \Phi_j^{a_j} = \Phi_j^{a_j} \Phi_i^{a_i} = 0 \), i.e \( f \) cannot be compatible. Therefore, the only monomials that appear in \( f \) are of the form \( x_i^{a_i} \) for \( i = 1, \ldots, n \), i.e \( f \) is in the image of the mapping

\[
\omega : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}[x_1, \ldots, x_n], \quad g \mapsto g(x_1, 0, \ldots, 0) + \cdots + g(0, \ldots, 0, x_n).
\]

(b)⇒(a): Any polynomial in the image of the mapping \( \omega \) is compatible by Remark 3.3.14.

Proposition 3.3.16. (Exclusion II) Let \( S = S(f_1, \ldots, f_r) \subseteq \mathbb{R}^n \) be a semialgebraic set and let \( S_{\text{nc}} = S(f_1, \ldots, f_r) \) be the corresponding non-commutative semialgebraic set. Then the following statement holds:

If the polynomials \( \text{Le}_z(f_1), \ldots, \text{Le}_z(f_r) \) are compatible for \( z \in \mathbb{Z}^n \), then

\[
S(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r))
\]

is bounded if and only if

\[
S(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r))
\]

is bounded.

Proof: ⇒: Let \( S(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)) \) be bounded. Since \( \text{Le}_z(f_1), \ldots, \text{Le}_z(f_r) \) is compatible, the polynomials are in the image of \( \omega \) according to Lemma 3.3.15. So we can write

\[
\text{Le}_z(f_1) = \sum_{s_1} a_{i_1} x_i^{s_1} \]

\[
\vdots
\]

\[
\text{Le}_z(f_r) = \sum_{s_r} a_{i_r} x_i^{s_r}.
\]

Let \( A_1, \ldots, A_n \in \text{Her}_k \) and \( \lambda_i(A_j) \) the \( i \)-th largest eigenvalue of \( A_j \). Let us fix some natural numbers \( i_1', \ldots, i_r', s_1, \ldots, s_r \) such that \( a_{i_1's_1}, \ldots, a_{i_r's_r} \neq 0 \). Using [7, Corollary III.2.2, p. 63] we get that

\[
\lambda_j(\text{Le}_z(f_1)(A_1, \ldots, A_n)) \leq a_{i_1's_1} \lambda_j \left( A_{i_1's_1} \right) + \sum_{(i_1, s_1) \neq (i_1', s_1)} a_{i_1's_1} \lambda_1 \left( A_{i_1's_1} \right) \]

\[
\vdots
\]

\[
\lambda_j(\text{Le}_z(f_r)(A_1, \ldots, A_n)) \leq a_{i_r's_r} \lambda_j \left( A_{i_r's_r} \right) + \sum_{(i_r, s_r) \neq (i_r', s_r)} a_{i_r's_r} \lambda_1 \left( A_{i_r's_r} \right).
\]

Now, suppose that one matrix, say \( A_1 \), becomes arbitrary large. Thus, there is some \( j \) such that \( |\lambda_j(A_1)| \) becomes arbitrary large. Let \( s_i = 1 \) when it is possible. Without loss of generality, we can assume that it is possible for at least one such \( s_i \). When \( |\lambda_j(A_1)| \) becomes arbitrary large there must be one inequality on the right hand side of (3.5) that becomes negative. Thus, the eigenvalues of \( A_1, \ldots, A_n \) must be bounded in order to have non-negativity. Since \( k \in \mathbb{N} \) was arbitrary we get that \( S(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)) \) is bounded.

⇐: This is trivial.
Example 3.3.17. Without the compatibility condition, almost boundedness does not hold as in Theorem 3.3.16. Take the polynomial \(-x_1^2x_2^2 - x_3^2x_4^2 \in \mathbb{C} \langle x_1, x_2 \rangle_{\text{her}}\) and consider \(S^{nc} = S(-x_1^2x_2^2 - x_3^2x_4^2)\). Then \(S^{nc}_1\) is almost bounded. We will show that \(c(S^{nc}) = \infty\).

Let \(MT_1, MT_2 \in MT^+(S^{nc})\). Take two fibers \(F_1\) and \(F_2\) of \(MT_1\) and \(MT_2\). Let \(0 \in \mathbb{R}^{2(k \times k)}\) and \(0' \in \mathbb{R}^{2(k' \times k)}\) be matrices where all entries vanish. Define new fibers \(F'_1\) and \(F'_2\) of \(MT_1\) resp. \(MT_2\) by setting

\[
(F'_1)_k = \begin{cases} 0 \oplus (F_1)_j & \text{if } k = 2j \\
(F_1)_k & \text{else} 
\end{cases}
\]

and

\[
(F'_2)_k = \begin{cases} (F_2)_j \oplus 0 & \text{if } k = 2j \\
(F_2)_k & \text{else} 
\end{cases}
\]

From \(F'_1\) and \(F'_2\) we get a third fiber of some new tentacle \(MT'\) that has the first component from \(F'_1\) and the second from \(F'_2\). Taking the defining functions of this new tentacle at the 2\(k\)-th level shows that they do not form a compatible tuple and that \(c(S^{nc}) > 0\).

By using this kind of argument one can even achieve \(c(S^{nc}) = \infty\). Note that the same argument does not work with any polynomials if we replace almost bounded by bounded.

Theorem 3.3.18. (Exclusion III) Let \(S = S(f_1, \ldots, f_r) \subseteq \mathbb{R}^n\), where \(\deg(f_i) > 0\) for \(i = 1, \ldots, r\). Let the following statement hold:

We have that

\[
S(\text{Le}(f_1), \ldots, \text{Le}(f_r))
\]

is \((n, r)\)-row compact.

Let \(V = \bigcup_k \text{Sym}_k^n \subseteq \mathcal{K}(\mathcal{H})^n\) and \(MT \in MT(V)\) such that for each \(k \in \mathbb{N}\) we have that \(\deg(MT_k)\) has one positive element that is strictly bigger than all the other entries. Then \(MT_k \notin MT(S^{nc}_k)\) for all \(k \in \mathbb{N}\).

**Proof:** In the following let \(z = \max(\deg(MT_k))\).

Step I: Rephrasing the problem: Let \(MT_k\) be given by \(\Phi_1, \ldots, \Phi_n\) with basis \(B\). Take a fiber \(F = (\text{Fibre}_{A, k}(MT_k))^k\). Furthermore, let the polynomials \(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)\) be given by

\[
\text{Le}_z(f_i) = \sum_{\alpha} \left( \frac{1}{2} a_{\alpha i} x_1^{\alpha_i} \cdots x_n^{\alpha_n} + \frac{1}{2} a_{\alpha i} x_2^{\alpha_i} \cdots x_1^{\alpha_n} \right) \neq 0
\]

for \(i = 1, \ldots, r\). Restricting the polynomials \(\text{Le}_z(f_1), \ldots, \text{Le}_z(f_r)\) onto the fiber \(F\) leads to

\[
\sum_{\alpha} \left( \frac{1}{2} a_{\alpha i} (\Phi(t) \odot A_i)^{\alpha_i} \cdots (\Phi_n(t) \odot A_n)^{\alpha_n} + \frac{1}{2} a_{\alpha i} (\Phi_n(t) \odot A_n)^{\alpha_i} \cdots (\Phi(t) \odot A_1)^{\alpha_n} \right)
\]

for \(A_1, \ldots, A_n \in S^{nc}_k\). We will show that there is a number \(l \in \mathbb{N}\) such that

\[
\left( \sum_{\alpha} \left( \frac{1}{2} a_{\alpha i} (\Phi(t) \odot A_1)^{\alpha_i} \cdots (\Phi_n(t) \odot A_n)^{\alpha_n} + \frac{1}{2} a_{\alpha i} (\Phi_n(t) \odot A_n)^{\alpha_i} \cdots (\Phi(t) \odot A_1)^{\alpha_n} \right) \right)_l \to -\infty
\]

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as \( t \to \infty \).

Step II: Using the row compactness: We know that \( S(\text{Le}(f_1), \ldots, \text{Le}(f_r)) \) is row-compact. This implies that
\[
S(\omega(\text{Le}(f_1)), \ldots, \omega(\text{Le}(f_r)))
\]
is compact. Let \( z_j \) be the biggest component of \( z \). It is not difficult to see that there is an index \( i = 1, \ldots, r \) such that \( x_j^{\text{gr}(f_i)} \) appears in \( \text{Monomial}(\omega(\text{Le}(f_i))) \) and therefore in \( \text{Monomial}(\text{Le}(f_i)) \).

Step III: Proving the convergence: Take \( i = 1, \ldots, r \) such that \( \omega(\text{Le}(f_i)) \neq 0 \), i.e. \( x_j^{\text{gr}(f_i)} \in \text{Monomial}(\text{Le}(f_i)) \). As pointed out in step II, the compactness of
\[
S(\omega(\text{Le}(f_1)), \ldots, \omega(\text{Le}(f_r)))
\]
implies that we can choose \( j \) such that \( z_j \) is the biggest entry of \( z \). The monomial \( x_j^{\text{gr}(f_i)} \) has also the biggest \( z \)-degree of all monomials of \( f_i \) and therefore \( x_j^{\text{gr}(f_i)} \in \text{Monomial}(\text{Le}_z(f_i)) \). In fact, the \( z \)-degree of \( x_j^{\text{gr}(f_i)} \) is strictly bigger than the \( z \)-degree of any other monomial of \( \text{Le}_z(f_i) \). Since
\[
S(\omega(\text{Le}(f_1)), \ldots, \omega(\text{Le}(f_r)))
\]
is compact, we can assume that \( \text{Term}(x_j^{\text{gr}(f_i)}) = ax_j^{\text{gr}(f_i)} \) satisfies
\[
a x_j^{\text{gr}(f_i)} \to -\infty
\]
as \( x_j \to \infty \). By assumption \( z_j \) is the biggest component of \( z \) and therefore the (strictly) biggest entry of \( \text{deg}(\mathcal{M}_k) \) is in \( \text{deg}(\Phi_j) \). Let \( \text{deg}(\Phi_j)_{l_l} = z_j \) for some index \( l = 1, \ldots, k \).

By choosing an appropriate fibre \( F \), we get
\[
(\text{a}(\Phi_j(t) \odot A_j)^{\text{gr}(f_i)})_{l_l} \to -\infty
\]
for \( t \to \infty \). Since \( z_j \) was strictly the biggest entry of \( z \) and by the choice of \( l \), we get that
\[
\left( \sum_{\alpha_1} \left( \frac{1}{2} a_{\alpha_1} (\Phi_1(t) \odot A_1)^{\alpha_1} \cdots (\Phi_n(t) \odot A_n)^{\alpha_n} + \frac{1}{2} a_{\alpha_1} (\Phi_1(t) \odot A_1)^{\alpha_1} \cdots (\Phi_n(t) \odot A_n)^{\alpha_n} \right) \right)_{l_l} \to -\infty
\]
as \( t \to \infty \).

3.4 Semialgebraic sets, joint spectra and operator systems

In the previous section we were concerned about the geometry of non-commutative semialgebraic sets with respect to objects we called tentacles. Theorem 3.3.8 is a good example how difficult it is to conclude something about a non-commutative semialgebraic
set that comes from a commutative semialgebraic set. Even properties like compactness are only preserved under very strict conditions as Proposition 3.3.16 and Example 3.3.17 illustrate it.

In the following we want to use the notion of a joint spectrum of a tuple of bounded operators to get more information about the geometry of $S^{sc}$. The joint spectrum by R. Harte [23] is defined in the following way:

**Definition 3.4.1.** Let $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$ be a tuple of bounded operators on some Hilbert space $\mathcal{H}$. The joint spectrum $\sigma_{\mathcal{H}}(T_1, \ldots, T_n)$ of $(T_1, \ldots, T_n)$ is defined to be the set of all points $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$ such that $(\lambda_1 - T_1, \ldots, \lambda_n - T_n)$ generates a proper left or right ideal of $B(\mathcal{H})$.

The joint spectrum has some useful properties:

**Lemma 3.4.2.** Let $(T_1, \ldots, T_n) \in B(\mathcal{H})^n$. Then the following statements hold:

a. The set $\sigma_{\mathcal{H}}(T_1, \ldots, T_n)$ is contained in $\sigma(T_1) \times \cdots \times \sigma(T_n)$.

b. Let $f \in \mathbb{C}[x_1, \ldots, x_n]$. Then $f(\sigma_{\mathcal{H}}(T_1, \ldots, T_n)) \subseteq \sigma_{\mathcal{H}}(f(T_1, \ldots, T_n))$.

c. If the operators $T_1, \ldots, T_n$ are self-adjoint, then

$$
\sigma_{\mathcal{H}}(T_1, \ldots, T_n) = \left\{ (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n : \inf_{\|x\|=1} \sum_j \| (T_j - \lambda_j)x \| = 0 \right\}.
$$

**Proof:** (a): That is obvious.


(c): See [46] Proposition 1, p. 165.

**Proposition 3.4.3.** Let $f_1, \ldots, f_r \in \mathbb{C}[x_1, \ldots, x_n]$ be such that $S(f_1, \ldots, f_r)$ is compact and let $\mathcal{E} = \{(T_1, \ldots, T_n) \in K(\mathcal{H})^n : \sigma_{\mathcal{H}}(T_1, \ldots, T_n) \neq \emptyset \}$. Then $S(f_1, \ldots, f_r) \cap \mathcal{E}$ is almost bounded.

**Proof:** According to Lemma 3.4.2 (b) we have

$$
S(f_1, \ldots, f_r) \cap \mathcal{E} = \{(T_1, \ldots, T_n) \in \mathcal{E} : \sigma_{\mathcal{H}}(f_1(T_1, \ldots, T_n)), \ldots, \sigma_{\mathcal{H}}(f_r(T_1, \ldots, T_n)) \subseteq [0, \infty) \}
\subseteq \{(T_1, \ldots, T_n) \in \mathcal{E} : f_1(\sigma_{\mathcal{H}}(T_1, \ldots, T_n)), \ldots, f_r(\sigma_{\mathcal{H}}(T_1, \ldots, T_n)) \subseteq [0, \infty) \}
= \{(T_1, \ldots, T_n) \in \mathcal{E} : \sigma_{\mathcal{H}}(T_1, \ldots, T_n) \subseteq S(f_1, \ldots, f_r) \}.
$$

It is not difficult to see that $S(f_1, \ldots, f_r) \cap \mathcal{E}$ is almost bounded.

**Remark 3.4.4.** One can identify the Grassmanian of $\mathcal{H}$ with the set of all projections onto a closed subspace of $\mathcal{H}$. I.e we have

$$
\mathcal{G}(\mathcal{H}) = \{ P \in B(\mathcal{H}) : P \text{ is an orthogonal projection} \}.
$$
According to [3, p. 42] we have the following decomposition of $B(\mathcal{H})_{\text{her}}$: Fix a projection $P \in \mathcal{G}(\mathcal{H})$. Then any element $A \in B(\mathcal{H})$ can be written as

$$A = \begin{pmatrix} PA P & PA P^\perp \\ P^\perp AP & P^\perp AP^\perp \end{pmatrix}.$$ 

Thus, every operator $A$ decomposes into a diagonal and one co-diagonal part. Let $\mathcal{D}_P$ be the set of all self-adjoint diagonal operators with respect to $P$ and let $C_P$ be the set of all self-adjoint co-diagonal operators with respect to $P$. Then

$$B(\mathcal{H})_{\text{her}} = \mathcal{D}_P \oplus C_P.$$ 

Let $P_1$ be an orthogonal projection onto a one-dimensional subspace of $\mathcal{H}$. If $S(f_1, \ldots, f_r)$ is a compact semi-algebraic set in $\mathbb{R}^n$, then $S(f_1, \ldots, f_r) \cap \mathcal{D}_{P_1}$ is bounded.

**Remark 3.4.5.** Let $T_1, \ldots, T_n \in K(\mathcal{H})_{\text{her}}$. By [45, Proposition 1, p. 165] the joint spectrum $\sigma_H(T_1, \ldots, T_n)$ equals the joint spectrum $\sigma_H(T_1, \ldots, T_n)$ in the Banach algebra $B(T_1, \ldots, T_n)$ generated by $T_1, \ldots, T_n$ and the identity. By [37, Theorem 1, p. 88] $\sigma_H(T_1, \ldots, T_n) = \emptyset$ if only if the two-sided ideal generated by all the commutators equals the whole algebra $B(T_1, \ldots, T_n)$. Proposition 3.4.3 implies that the set of all elements $(T_1, \ldots, T_n) \in S^{\text{sc}}$, for which the Banach algebra $B(T_1, \ldots, T_n)$ has the property that the two-sided ideal generated by all the commutators is proper, is almost bounded.

**Remark 3.4.6.** Let $C \subseteq \mathbb{R}^n$ be a convex salient cone and consider the sets

$$J(C)_k := \{(T_1, \ldots, T_n) \in \text{Her}_n^\mathbb{C} : \sigma_H(T_1, \ldots, T_n) \subseteq C\}$$

for $k \in \mathbb{N}$ and

$$J(C) = \bigcup_k J(C)_k.$$ 

Then $J(C)$ has the following properties:

a If $A, B \in J(C)$, then $A \oplus B = (A_1 \oplus B_1, \ldots, A_n \oplus B_n) \in J(C)$: By using Lemma 3.4.2(c) it is easy to see that

$$\sigma_H(A_1 \oplus B_1, \ldots, A_n \oplus B_n) = \sigma_H(A_1, \ldots, A_n) \cup \sigma_H(B_1, \ldots, B_n).$$

Since $\sigma_H(A_1, \ldots, A_n) \subseteq C$ and $\sigma_H(B_1, \ldots, B_n) \subseteq C$, the union must also be contained in $C$. Thus $A \oplus B \in J(C)$.

b If $A \in J(C)$, then $U^* AU = (U^* A_1 U, \ldots, U^* A_n U) \in J(C)$ for every unitary $U \in U_1$ of the right size $k$: We have

$$\inf_{\|x\|=1} \sum_j \|U^* A_j U - \lambda_j x\| = \inf_{\|x\|=1} \sum_j \|U^* A_j U - \lambda_j x\| = \inf_{\|x\|=1} \sum_j \|U^* A_j U - \lambda_j x\| = \inf_{\|x\|=1} \sum_j \|U^* A_j U - \lambda_j U^* x\| = \inf_{\|x\|=1} \sum_j \|U^* A_j U - \lambda_j U^* x\| = \inf_{\|x\|=1} \sum_j \|A_j x - \lambda_j x\| = 0$$

and Lemma 3.4.2(c) implies the assertion.
In general, the set $J(C)_2$ is not convex. We can restrict ourselves to the two dimensional case, i.e $C \subseteq \mathbb{R}^2$. Furthermore, we will assume that $C = \mathbb{R}_{\geq 0}(1,0)$. Then $J(C)_2$ can be described as follows: It consists of all $(A_1,A_2) \in \text{Her}_2^2$ such that $A_1$ and $A_2$ have no common eigenvector or if the have one, then the corresponding eigenvalue with respect to $A_2$ is 0 and non-negative with respect to $A_1$.

Now, $J(C)_2$ is not convex. Take $(B_1,B_2) \in \text{Her}_2^2$ such that $\sigma_H(B_1,B_2) = \emptyset$, i.e $B_1$ and $B_2$ have no common eigenvectors, with $B_2 - B_1 \succeq 0$. Then $\sigma_H(B_2 - B_1,0) \in C$. The connecting line $\{(t(B_2 - B_1,0) + (1 - t)(B_1,B_2)) : t \in [0,1]\}$ contains $(\frac{1}{2}B_2,\frac{1}{2}B_2)$. But $B_2 \neq 0$ and therefore $\sigma_H(\frac{1}{2}B_2,\frac{1}{2}B_2) \notin C$.

What we know now is, that $J(C)$ does not need to be a matrix-convex set.

Let us revisit $J(C)_2$. For two distinct points $A = (A_1,A_2), B = (B_1,B_2)$ we define the polynomial

$$f_{A,B} = \det ([(1 - t)A_1 + tB_1,(1 - t)A_2 + tB_2]) \in \mathbb{R}[t],$$

where $[\cdot,\cdot]$ denote the commutator brackets. Let $t' \in [0,1]$ with $f_{A,B}(t') \neq 0$ and consider the corresponding point $(T_{1,t},T_{2,t'}) = ((1 - t)A_1 + tB_1,(1 - t)A_2 + tB_2)$. By Remark 3.4.5 and [6], Theorem 3, p. 2 we get the following implications:

The two matrices $T_{1,t} \in M_2(\mathbb{C})$ and $T_{2,t'} \in M_2(\mathbb{C})$ generate $M_2(\mathbb{C}) \Leftrightarrow \sigma_H(T_{1,t},T_{2,t'}) = \emptyset$. Furthermore, the above implications lead straight to the conclusion that $J(C)_2$ is dense in $\text{Her}_2^2$.

This problem motivates the following approach that can be found in [54]: Let $T_1,\ldots,T_n$ be elements that are contained in the center of a complex algebra $A$ and let $B$ be a left $A$-module. In addition, let $W$ be a $k$-dimensional $\mathbb{C}$-vector space with basis $w_1,\ldots,w_k$. Then we set $\wedge W$ to be anti-commutative graded vector space $\bigoplus_i \wedge^i W$. The vector space $B \otimes_\mathbb{C} \wedge W$ inherits this structure by the canonical isomorphism

$$B \otimes_\mathbb{C} \wedge^i W \cong \wedge^i (B \otimes_\mathbb{C} W).$$

For each $i$ we have a homomorphism $d : B \otimes \wedge^i W \to B \otimes \wedge^{i-1} W$ given by sending $B \otimes w_1 \wedge \cdots \wedge w_i$ to $\sum_{j=1}^i (-1)^{j-1}T_j B \otimes w_{i+1} \wedge \cdots \wedge w_i \wedge w_j$. Using the assumption that the operators $T_1,\ldots,T_n$ mutually commute such that $d^2 = 0$. In other words, we get a complex

$$\cdots \to B \otimes \wedge^i W \to B \otimes \wedge^{i-1} W \to \cdots \to B \to 0.$$

This complex is denoted with $\text{Kos}(T_1,\ldots,T_n)$ and is called the KOSZUL COMPLEX for $T_1,\ldots,T_n$. The TAYLOR JOINT SPECTRUM is defined to be the set

$$\sigma_T(T_1,\ldots,T_n) = \{(\lambda_1,\ldots,\lambda_n) \in \mathbb{C}^n : \exists k \geq 0 : H_k(\text{Kos}(\lambda_1 - T_1,\ldots,\lambda_n - T_n)) \neq 0\}.$$

This joint spectrum has the following properties:
i If \( T_1, \ldots, T_n \in B(\mathcal{H}) \) are commuting self-adjoint operators, then

\[
\sigma_T(T_1, \ldots, T_n) \subseteq \sigma_H(T_1, \ldots, T_n).
\]

This is a consequence of [54, Lemma 1.1, p. 177] and Lemma 3.4.2.

ii Let

\[
0 \to B_1 \to B_2 \to B_3 \to 0
\]

be a short exact sequence of \( A \)-modules. Then any joint spectrum of \( T_1, \ldots, T_n \) with respect to one of the modules \( B_1, B_2, B_3 \) is contained in the union of the other two joint spectra of \( T_1, \ldots, T_n \). This follows from [54, Lemma 1.2, p. 177].

iii The spectrum \( \sigma_T(T_1, \ldots, T_n) \) is compact. This is [54, Theorem 3.1, p. 185].

iv Let \( T_1^i, \ldots, T_n^i \in A \) be in the center. Then the spectrum \( \sigma_T(T_1^i, \ldots, T_n^i) \) is not empty. See [54, Corollary, p. 186].

v Let \( T_1, \ldots, T_n \in B(\mathcal{H}) \) and set \( T_i = \text{id}_H \otimes \cdots \otimes \text{id}_H \otimes T_i \otimes \cdots \otimes \text{id}_H \in B(\mathcal{H}^\otimes n) \). Then

\[
\sigma_T(T_1, \ldots, T_n) = \sigma_T(T_1 \otimes \cdots \otimes T_n) = \sigma(T_1) \times \cdots \times \sigma(T_n).
\]

See [12, pp. 307-310].

Property (v) motivates to define the following:

Let \( \mathcal{H} \) be a Hilbert space and consider the embeddings

\[
\begin{align*}
\iota_1 : B(\mathcal{H}) \hookrightarrow B(\mathcal{H}^\otimes n), & \quad A \mapsto A \otimes \text{id}_H \otimes \cdots \otimes \text{id}_H \\
\vdots & \\
\iota_n : B(\mathcal{H}) \hookrightarrow B(\mathcal{H}^\otimes n), & \quad A \mapsto \text{id}_H \otimes \cdots \otimes \text{id}_H \otimes A,
\end{align*}
\]

where \( \mathcal{H}^\otimes n \) denotes the \( n \)-fold tensor product. This motivates to define

\[
\tilde{J}(C)_k = \{ (A_1, \ldots, A_n) \in \text{Her}^n : \sigma_T(\iota_1(A_1), \ldots, \iota_n(A_n)) \subseteq C \}
\]

resp.

\[
\tilde{J}(C) = \bigcup_k \tilde{J}(C)_k
\]

for some convex salient cone \( C \subseteq \mathbb{R}^n \) that is non-empty and symmetric. Then \( \tilde{J} \) has the following properties:

A There are \( A, B \in \tilde{J}(C) \) such that \( A \not\oplus B \notin \tilde{J}(C) \): This follows straight from statement (v). For further explanation let us assume that the elements of the tuples \( A \) resp. \( B \) commute. In general, property (ii) tells us that the statement
depends on the choice of the $A$-module $B$. Let $A$ be the complex algebra that is generated by the tuples $A \otimes B$. Let $B = B(\mathcal{H}) \oplus B(\mathcal{H})$. We have an exact sequence

$$0 \to B(\mathcal{H}) \to B \to B(\mathcal{H}) \to 0.$$ 

The joint spectrum of $A \oplus B$ with respect to $B$ is contained in the joint spectrum of $A \oplus B$ with respect to $B(\mathcal{H})$. Using the fact that homology commutes with direct sums we can conclude that the joint spectrum of $A \oplus B$ with respect to $B$ is contained in $C$, if the joint spectra of $A$ and $B$ are contained in $C$. Note, that we had a similar statement for the Harte joint spectrum at the beginning of this remark.

B For each $s \geq 1$ the set $\mathcal{J}(C)_s$ is not a cone: Consider $\mathcal{J}(C)_s$, with respect to the cone

$$C = \text{conv}(\{(x_1, x_2) \in \mathbb{R}^2 : x_2 = \vert x_1 \vert \}).$$

(3.6)

Then we can use [27 Proposition 3, p. 5] to construct $(A_1, A_2), (B_1, B_2) \in \mathcal{J}(C)$, with $(A_1 + B_1, A_2 + B_2) \notin \mathcal{J}(C)_s$.

C If $A = (A_1, \ldots, A_n) \in \mathcal{J}(C)_s$, then $V^*AV \in \mathcal{J}(C)_s$, whenever $V \in M_{r,n}(\mathbb{C})$ with $V^*V = I$. Since $V^*V = I$, we have $\sigma_T(\lambda_1(A_1V), \ldots, \lambda_n(V^*A_nV)) \subseteq \sigma_T(\lambda_1(A_1), \ldots, \lambda_n(A_n))$.

Another way to see that $\mathcal{J}(C)$ is not an abstract operator system is the following one: Consider the operator system $(C^\text{min}_{2s})_{s \geq 0}$, defined in [22 Proposition 3, p. 8], given by

$$C^\text{min}_{2s} = \left\{ \sum_i c_i \otimes P_i : c_i \in C, P_i \in \text{Her}_{s}, P_i \geq 0 \right\}.$$ 

If $C$ is the cone from [3.6] then $\mathcal{J}(C)$ is not an abstract operator system: By construction we have $\mathcal{J}(C)_1 = C^\text{min}_{1} = C$. If $\mathcal{J}(C)_1$ is an abstract operator system, then $C^\text{min}_{2s} \subseteq \mathcal{J}(C)_s$. Suppose that $s > 1$ and consider the set $\{(1,1) \otimes P : P \geq 0\} \subseteq C^\text{min}_{2s}$. By assumption, we must have $\{(1,1) \otimes P : P \geq 0\} \subseteq \mathcal{J}(C)_s$. However, we can choose $P \geq 0$ such that the minimal eigenvalue $\lambda_{\text{min}}(P)$ of $P$ is strictly smaller than the maximal eigenvalue $\lambda_{\text{max}}(P)$ of $P$. Now, $(\lambda_{\text{max}}(P), \lambda_{\text{min}}(P)) \notin C$, resulting in a contradiction. Thus, $\mathcal{J}(C)$ is not an abstract operator system.

Consider the operator system $(C^\text{max}_{s})_{s \geq 0}$ defined in [22 Proposition 3, p. 5], i.e

$$C^\text{max}_{s} = \{(A_1, \ldots, A_n) \in \text{Her}_{s}^n : \forall v \in \mathbb{C} : (v^*A_1v, \ldots, v^*A_nv) \in C \},$$

where $C$ is again salient, convex, non-empty and symmetric. Then $\mathcal{J}(C)_s \subseteq C^\text{max}_{s}$ for all $s \geq 1$: We have the following inequalities

$$\lambda_{\text{min}}(A_1) \leq v^*A_1v \leq \lambda_{\text{max}}(A_1)$$

$$\vdots$$

$$\lambda_{\text{min}}(A_n) \leq v^*A_nv \leq \lambda_{\text{max}}(A_n)$$

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for all \( v \in \mathbb{C}^s \) with \( \|v\| = 1 \). If \( (A_1, \ldots, A_n) \in \tilde{J}(C)_s \), then

\[
\text{conv}(\sigma_T(\ell_1(A_1), \ldots, \ell_n(A_n))) = \prod_i [\lambda_{\min}(A_i), \lambda_{\max}(A_i)],
\]

implying \( \prod_i [\lambda_{\min}(A_i), \lambda_{\max}(A_i)] \subseteq C \). Using the above inequalities leads to the conclusion that the image of the map \( \mathbb{C}^s \to \mathbb{R}^n, v \mapsto (v^*A_1v, \ldots, v^*A_nv) \) is contained in \( C \). The situation is now as follows:

\[
\begin{align*}
\mathbb{C}^s_{\max} & \quad \subseteq \quad \mathbb{C}^s_{\min} \\
m\text{conv}(\tilde{J}(C))_s & \quad \supseteq \quad \tilde{J}(C)_s \\
 & \quad \subset \quad ? \quad ?
\end{align*}
\]

The question mark in \( ? \) emphasizes that we do not know if we have an inclusion of sets. As an example we will consider the case where \( C \) is the cone generated by \((-1,1,1),(1,1,1),(1,-1,1),(-1,-1,1)\). Then we have \( \text{mconv}(\tilde{J}(C)) \neq C_{\min} \): We have

\[
C_{\min}^s = \{(-1,1,1)\otimes P_1 + (1,1,1)\otimes P_2 + (1,-1,1)\otimes P_3 + (-1,-1,1)\otimes P_4 : P_1, P_2, P_3, P_4 \in \text{Her}_s^+ \}.
\]

Applying the trace leads to

\[
\begin{align*}
& -\text{tr}(P_1) + \text{tr}(P_2) + \text{tr}(P_3) - \text{tr}(P_4) = x_1 \\
& \text{tr}(P_1) + \text{tr}(P_2) - \text{tr}(P_3) - \text{tr}(P_4) = x_2 \\
& \text{tr}(P_1) + \text{tr}(P_2) + \text{tr}(P_3) + \text{tr}(P_4) = x_3 \\
& \text{tr}(P_1) \geq 0 \\
& \text{tr}(P_2) \geq 0 \\
& \text{tr}(P_3) \geq 0 \\
& \text{tr}(P_4) \geq 0
\end{align*}
\]

with \( x_1, x_2, x_3 \in \mathbb{R} \). By taking some appropriate elements in \( \partial \tilde{J}(C)_s \) we see that there must be a solution to

\[
\begin{align*}
& -\text{tr}(P_1) + \text{tr}(P_2) + \text{tr}(P_3) - \text{tr}(P_4) = -2 \\
& \text{tr}(P_1) + \text{tr}(P_2) - \text{tr}(P_3) - \text{tr}(P_4) = 0 \\
& \text{tr}(P_1) + \text{tr}(P_2) + \text{tr}(P_3) + \text{tr}(P_4) = 2 \\
& \text{tr}(P_1) \geq 0 \\
& \text{tr}(P_2) \geq 0 \\
& \text{tr}(P_3) \geq 0 \\
& \text{tr}(P_4) \geq 0
\end{align*}
\]
in order to have $\text{mconv}(\mathcal{J}(\mathcal{C})) = \mathcal{C}^{\min}$. From the above equations we get

\[ \begin{align*}
\text{tr}(P_2) &= 1 - \text{tr}(P_1) \\
\text{tr}(P_3) &= -1 + \text{tr}(P_1) \\
\text{tr}(P_4) &= 2 - \text{tr}(P_1).
\end{align*} \]

Since $\text{tr}(P_1), \text{tr}(P_2), \text{tr}(P_3), \text{tr}(P_4) \geq 0$, we get $\text{tr}(P_1) = \text{tr}(P_3) = 1$ and $\text{tr}(P_2) = \text{tr}(P_4) = 0$. Consider

\[ \left( \begin{array}{ccc}
-1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{array} \right), \left( \begin{array}{ccc}
0 & 1 \\
1 & 0 \\
0 & 1
\end{array} \right) \in \mathcal{J}(\mathcal{C})_2 \]

and the corresponding equations

\[ \begin{align*}
-P_1 + P_2 + P_3 - P_4 &= \left( \begin{array}{ccc}
-1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{array} \right) \\
P_1 + P_2 - P_3 - P_4 &= \left( \begin{array}{ccc}
0 & 1 \\
1 & 0 \\
1 & 0 \\
0 & 1
\end{array} \right) \\
P_1 + P_2 + P_3 + P_4 &= \left( \begin{array}{ccc}
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{array} \right)
\end{align*} \]

\[ P_1 \succeq 0, \quad P_2 \succeq 0, \quad P_3 \succeq 0, \quad P_4 \succeq 0. \]

Let $x_1 = (P_1)_{11}, x_2 = (P_1)_{22}, x_3 = (P_2)_{11}, x_4 = (P_2)_{12}, x_5 = (P_3)_{11}, x_6 = (P_3)_{22}$. Consider the cone $\mathcal{C}_{\delta, \lambda}$ generated by $(-\delta, \lambda, 1), (\delta, \lambda, 1), (\delta, -\lambda, 1), (-\delta, -\lambda, 1)$ for $\lambda > 0$. Combining the trace equations with the equations above we get

\[ \begin{pmatrix}
-\delta & 0 & \delta & -\delta & 0 \\
0 & -\delta & -\delta & 0 & -\delta \\
\lambda & 0 & \lambda & -\lambda & 0 \\
0 & \lambda & -\lambda & 0 & -\lambda \\
1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1
\end{pmatrix} \begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5 \\
x_6
\end{pmatrix} = \begin{pmatrix}
-1 \\
1 \\
0 \\
0 \\
1 \\
1
\end{pmatrix}, \quad (3.7) \]

Suppose that $|\delta| = 1$. For $\delta = 1$ the system $[3.7]$ has no solution and for $\delta = -1$ we get that $\text{sign}(x_2) \neq \text{sign}(x_6)$. But then one of the matrices $P_1$ or $P_4$ is not positive semi-definite. This shows that

\[ \left( \begin{array}{ccc}
-1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1
\end{array} \right), \left( \begin{array}{ccc}
0 & 1 \\
1 & 0 \\
0 & 1
\end{array} \right) \in \mathcal{J}(\mathcal{C}_{\lambda})_2 \]

is not contained in $\mathcal{C}_{\delta, \lambda_2}^{\min}$ and therefore $\text{mconv}(\mathcal{J}(\mathcal{C}_{\delta, \lambda}))_2 \neq \mathcal{C}_{\delta, \lambda_2}^{\min}$.

The usage of $\mathcal{J}(\mathcal{C})$ together with the trace allowed us to construct a linear system of equations that imply $\text{mconv}(\mathcal{J}(\mathcal{C}))_2 \neq \mathcal{C}_2^{\min}$ and therefore $\mathcal{C}_2^{\min} \neq \mathcal{C}_2^{\max}$. 

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The definition of $\tilde{J}(C)$ of by using the Taylor joint spectrum involves homology. So there is an algebro-geometric relationship between the geometry of a cone, the minimal operator system and $\text{mconv}(\tilde{J}(C))$. A far more detailed description of these relationships can be found in [42].

**Theorem 3.4.7.** Let $C \subseteq \mathbb{R}^n$ be a salient polyhedral cone. Then $C_{\min} = C_{\max}$ if and only if $C$ is a simplicial cone. If $C$ is not simplicial, then $C_{2\min} \neq C_{2\max}$.

**Proof:** The first assertion is Proposition [22, Theorem 4.7, p. 10]. To verify the second statement we refine the proof of Proposition [22, Theorem 4.7, p. 10] in the following way:

**Step I:** The quadrilateral cone $C \subseteq \mathbb{R}^3$: By [42, Lemma 3.1, p. 10] the inclusion $C_{\max} \subseteq C_{\min}$ is invariant under affine transformation and therefore we can assume that $C$ is the cone generated by $(-1,1,1),(1,1,1),(1,-1,1),(-1,-1,1)$. In the above Remark 3.4.6 we showed that $\text{mconv}(\tilde{J}(C))_2$ does not equal $C_{\min}$, which implies $C_{2\min} \neq C_{2\max}$.

**Step II:** General cone $C' \subseteq \mathbb{R}^3$: We use a slightly different argumentation than the second step of Proposition [22, Theorem 4.7, p. 10]. Instead of considering the cone $C'$ that is given by an ellipse around the quadrilateral we proceed as follows: Let $C'$ be a non-simplicial cone. By applying an affine transformation, we can assume that there is a $\lambda > 0$ such that $C' \subseteq C_{\delta,\lambda}$ with $|\delta| = 1$. Thus, we can assume that

$$\left(\begin{array}{ccc}
-1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{array}\right), \left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{array}\right) \in \tilde{J}(C')_2.$$

By Remark 3.4.6 we have that

$$\left(\begin{array}{ccc}
-1 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
\end{array}\right), \left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0 \\
\end{array}\right) \notin C_{\delta,\lambda_{2\min}}.$$

The inclusion $C_{2\min} \subseteq C_{\delta,\lambda_{2\min}}$ implies the assertion.

**Step III:** Arbitrary cone $C \subseteq \mathbb{R}^n$: It is not difficult to adjust the third step of Proposition [22, Theorem 4.7, p. 10] to work here as well. □
Remark 3.4.8. For all the technical details of this remark we refer to \[51\].

Let \( T_1, \ldots, T_n \) be a collection of operators on some Hilbert space \( \mathcal{H} \). For a polynomial \( f \in \mathbb{R}(x_1, \ldots, x_n) \) we usually do not have \( f(\sigma(T_1, \ldots, T_n)) = \sigma(f(T_1, \ldots, T_n)) \). According to [23], p. 874 equality holds for example if \( T_1, \ldots, T_n \) are normal bounded operators that commute with each other. Here we are interested in the case where \( f(\sigma(T_1, \ldots, T_n)) \neq \sigma(f(T_1, \ldots, T_n)) \) and how large the difference between these sets can become. In order to do that, let us consider the following situation: Let

\[
 f = \sum_{|\alpha|=2k} a_\alpha x^\alpha
\]

be homogenous polynomials such that \( S(f) = \{0\} \) and

\[
 (-1)^k \sum_{|\alpha|=2k} a_\alpha x^\alpha > \mu |x|^{2k},
\]

\((k = \frac{\text{deg}(f)}{2})\) for all \( x \in \mathbb{R}^n \) and some \( \mu > 0 \). For each \( j = 1, \ldots, n \) let \( \partial_j \) the differential operator \( \partial_{x_j} \). Then the operator \( A := f(\partial_1, \ldots, \partial_n) \) is an \textsc{uniform elliptic partial differential operator}. We will now introduce the notion of \( L^2 \)-based Sobolev spaces. These spaces are defined through

\[
 W^s_2(D) := \left\{ g \in L^2(D) : \int_D \left( 1 + |y|^2 \right)^s |\mathcal{F}g(y)|^2 dy < \infty \right\},
\]

where \( \mathcal{F} \) denotes the Fourier-transform, \( D \subseteq \mathbb{R}^n \) is some bounded domain and \( s \in \mathbb{R} \).

In other words, \( W^s_2(D) \) is the closure of the space \( C_0^{\infty}(D) \) with respect to the norm

\[
 ||g||_{W^s_2} = \left( \int_D \left( 1 + |y|^2 \right)^s |\mathcal{F}g(y)|^2 dy \right)^{\frac{1}{2}}.
\]

Let \( D \) be bounded. By the \textsc{Garding inequality} there is \( C_\epsilon > 0 \) for any \( \epsilon > 0 \) such that

\[
 \langle Ag, g \rangle_{L^2} \geq (\mu - \epsilon) ||g||_{W^2_2}^2 - C_\epsilon ||g||_{L^2}^2
\]

for any \( g \in C_0^{\infty}(D) \). The operator \( \tilde{A} := A + \gamma \text{id} \) is non-negative whenever \( \gamma > C_\epsilon \) and \( \tilde{A} \) has a self-adjoint \textsc{Friedrichs-extension} \( \tilde{A}_F \) on \( W^{\frac{n}{2}}_2(D) \cap W^n_2(D) \). Every element of \( \sigma(\tilde{A}_F) \) is an eigenvalue of \( \tilde{A}_F \) and \( \sigma(\tilde{A}_F) \cap [0, \infty) \) is unbounded. In contrast to that the real non-negative part of \( f(\sigma(\partial_1) \times \cdots \times \sigma(\partial_n)) + \gamma \) is bounded. Note that \( \partial_1, \ldots, \partial_n \) are not self-adjoint operators. However, one can construct similar examples with \textsc{Dirac operators} which are essentially self-adjoint and satisfy the Garding-inequality as described in [37], 5.14, p. 76.

Example 3.4.9. Let \( f = -x_1^2 - \cdots - x_n^2 \). Then \( S(f) = \{0\} \), \( f \) is homogenous and

\[
 -(-x_1^2 - \cdots - x_n^2) > \mu |x|^2
\]

for all \( x \in \mathbb{R}^n \) and \( 0 < \mu < 1 \). Thus \( -\Delta = f(\partial_1, \ldots, \partial_n) \) is an \textsc{uniform elliptic operator} that is defined on \( W^n_2(D) \) for some bounded domain \( D \subseteq \mathbb{R}^n \). By Remark 3.4.8 we know that the extension \( (-\Delta + \gamma \text{id})_F \) has the property that \( \sigma((-\Delta + \gamma \text{id})_F) \cap [0, \infty) \) is unbounded. On the other hand side, the real part of \( f(\sigma(\partial_1), \ldots, \sigma(\partial_n)) + \gamma \) is bounded.
3.5 Non-commutative archimedean quadratic modules

In the commutative case the quadratic module $QM(-1)$ is stable but not totally stable. The stability follows from the identity $f = \left(\frac{f+1}{2}\right)^2 + \left(\frac{f-1}{2}\right)^2 (1)$ for all $f \in \mathbb{R}[x_1, \ldots, x_n]$. While it seems unclear if $QM(-x_1^2, \ldots, -x_r^2)$ is stable it is known that $M = QM(1 - x_1^2, \ldots, 1 - x_r^2)$ is not stable for $r \geq 2$. Suppose that $M$ is stable. By the Schmuedgen’s Positivstellensatz and by the closedness of $M$, we get that $M$ equals the set of all polynomials that are non-negative on the set $S(1 - x_1^2, \ldots, 1 - x_r^2)$. In other words, $M$ must be saturated. But this contradicts [35, Example 4.2.4 (iv), p.62].

In this section we will investigate the stability of these quadratic modules in the non-commutative sense. An obvious question would be, if $QM(h, x_1^2, \ldots, x_r^2)$ is stable for $r \geq 2$ or not. To answer this question and to continue our investigations, we need to know what residual finite-dimensional quadratic modules are and what the strong Positivstellensatz states. Both ingredients are explained in [2] and we will give a detailed description of them.

**Definition 3.5.1.** A quadratic module $M_{nc}$ of $\mathbb{C}[x_1, \ldots, x_n]$ is called **archimedean** if $n - f^*f \in M$ for any $f \in \mathbb{C}[x_1, \ldots, x_n]$ and large enough $n \in \mathbb{N}$.

**Lemma 3.5.2.** Each archimedean quadratic module $M$ gives rise to a $C^*$-algebra $C^*(\mathbb{C}[x_1, \ldots, x_n], M)$ which we call the universal $C^*$-algebra of $M$ with respect to $\mathbb{C}[x_1, \ldots, x_n]$.

**Proof:** See [2, Definition 2.6, p.4].

**Theorem 3.5.3.** Let $A$ be a $C^*$-algebra, $\pi \in \text{Sp}(A)$ and $Y \subseteq \text{Sp}(A)$. Then the following statements are equivalent:

a) $\pi$ is in the closure of $Y$.

b) The kernel of $\pi$ is contained in the set $\mathcal{V}(\text{ker}(Y))$.

c) Every positive state in $\text{St}(A)$ associated with $\pi$ is a weak $*$-limit of linear combinations of positive states associated with elements out of $S$.

d) Every state in $\text{St}(A)$ associated with $\pi$ is a weak $*$-limit of states associated with $Y$.

**Proof:** See [18, Theorem 3.4.10, p. 79] and [18, Proposition 3.4.2, p. 76].

**Definition 3.5.4.** An archimedean quadratic module $M_{nc}$ in $\mathbb{C}[x_1, \ldots, x_n]$ is called residual finite dimensional if

$$\mathcal{V}\left(\text{ker}\left(\bigcup_{n \in \mathbb{N}} \text{IrrRep}_n (C^*(\mathbb{C}[x_1, \ldots, x_n], M))\right)\right) = \text{Prim}(C^*(\mathbb{C}[x_1, \ldots, x_n], M)).$$
Remark 3.5.5. Definition 3.5.4 and Definition 2. Definition 3.3, p. 8 are equivalent. This follows from Theorem 3.5.3 and [2, Theorem 3.8, p. 9].

Proposition 3.5.6. Let $M$ be an archimedean quadratic module in $\mathbb{C}(x_1, \ldots, x_n)$. Then the following statements are equivalent:

a) The quadratic module $M^{nc}$ is residual finite dimensional.

b) If $f \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}}$ satisfies $\pi(f) \geq 0$ for all $\pi \in \lim_{k \to \infty} M^{nc}_k$, then $f + \varepsilon \in M^{nc}$ for all $\varepsilon > 0$.

Proof: See [2, Theorem 3.4, p. 8].

Definition 3.5.7. An archimedean quadratic module $M^{nc}$ in $\mathbb{C}(x_1, \ldots, x_n)$ is said to have the **strong Positivstellensatz property** if statement (b) of Proposition 3.5.6 remains true for $\varepsilon = 0$.

An archimedean quadratic module $M^{nc}$ in $\mathbb{C}(x_1, \ldots, x_n)$ is said to have the **compression property over** $\mathbb{R}$ if for every finite dimensional $*$-subspace $U$ of $\mathbb{C}(x_1, \ldots, x_n)$ we have that

$$M^{nc} \cap U = M^{nc}_k \cap U$$

for $M^{nc}_k = \{ f \in \mathbb{C}(x_1, \ldots, x_n)_{\text{her}} : \pi(f) \geq 0, \pi \in M^{nc}_k \}$ and $k \in \mathbb{N}$ big enough. If we replace $\mathbb{R}$ by any real closed extension field and the above statement remains true, then we say that $M^{nc}$ has the **compression property**.

Remark 3.5.8. It is not difficult to see that the compression property implies the strong Positivstellensatz property. However, it is not clear if the strong Positivstellensatz property implies the compression property.

Theorem 3.5.9. Suppose that the quadratic module $M^{nc} = QM(f_1, \ldots, f_r)$ has the compression property. Then $QM(f_1, \ldots, f_r)$ is stable.

Proof: Let $U$ be finite dimensional $*$-subspace of $\mathbb{C}(x_1, \ldots, x_n)$.

By assumption $M^{nc} \cap U$ is semialgebraic. Let $W_d$ be the $*$-subspace of $\mathbb{C}(x_1, \ldots, x_n)$ that contains all polynomials up to some degree $d \in \mathbb{N}$. Now, $M^{nc} \cap U$ is semialgebraic and $\cup_d M^{nc} \cap U$ is a semialgebraic covering of $M^{nc} \cap U$. If we extend to a $\mathbb{R}_1$-saturated real closed field $\mathbb{F} \supsetneq \mathbb{R}$, then the covering can be chosen to be finite by [45, 2.2.11, p. 40]. The transfer principle [45, 2.1.10, p. 35] implies $M^{nc} \cap U = M^{nc} \cap U$ for big $d \in \mathbb{N}$ over the original field $\mathbb{R}$. In other words, $M^{nc}$ is stable.

Example 3.5.10. The quadratic module $QM(1 - x_1^2, \ldots, 1 - x_n^2)$ is not totally stable, but stable. This follows from [2, Example 3.13, p. 11] and Theorem 3.5.9.

Remark 3.5.11. In our situation, Theorem 3.5.9 is at least as strong as the scalar valued convex Positivstellensatz [26, Theorem 1.1, p.5].
4 Stability of group rings

4.1 Introduction

Definition 4.1.1. Let $R$ be a ring with an involution $R \to R, x \mapsto x^*$, $G$ a group that is generated by some finite set $P$ and $z \in \mathbb{Z}^{|P|}$. Let $g_z : G \to \mathbb{Z}$ be the function that is defined in the following way: We assign to each $p \in P$ a number $z_p \in \mathbb{Z}$, where $z_p$ is the $p$-th component of $z$. Thus, to each combination of elements in $P$ we can assign a number in an obvious way by summing the corresponding $z_p$ up. For an element $g \in G$ we define $g_z(g)$ to be the smallest number that we can assign to a combination of elements of $P$ that express $g$. We call $g_z(g)$ the $z$-degree. If every component of $z$ equals 1, then we just write $g_z(g)$ instead of $g_z(g)$.

Note, that for any two $g_1, g_2 \in G$ we have

$$g_z(g_1 g_2) \leq g_z(g_1) + g_z(g_2).$$

Each of these functions $g_z$ gives rise to a QUASI $z$-GRADING of the group ring $R[G]$ by writing

$$R[G] = \bigoplus_{i \in \mathbb{Z}} \left\{ \sum_g r_g g : g \in G, k_g \in \mathbb{K}, g_z(g) = i \right\} := \bigoplus_{i \in \mathbb{Z}} R[G]_{i,z}.$$

Note the similarities between the above identification and the identification made in Definition 2.1.2. Again the LEADING TERM will be denoted with $Le_z$. There is an involution $*: R[G] \to R[G]$ given by $(\sum_g r_g g)^* = \sum_g r_g g^{-1}$. Elements invariant under $*$ are called hermitian and the set of all these elements will be denoted with $R[G]_{\text{her}}$. Finally, a quadratic module $QM(g_1, \ldots, g_n)$ generated by $g_1, \ldots, g_n \in R[G]_{\text{her}}$ is given by the identification

$$QM(g_1, \ldots, g_n) = \left\{ \sum_{i,j} q_{ij} g_i g_j + s : q_{ij} \in R[G], s \in \Sigma R[G]^2 \right\},$$

where $\Sigma R[G]^2 = \left\{ \sum_i q_i^2 : q_i \in R[G] \right\}$ is the set of sums of squares in $R[G]$. These definitions are in complete analogy to the definition made in 2.1.6. As in Definition 2.1.7 we say that $QM(g_1, \ldots, g_n)$ is totally stable if $g_z(f + g) \geq g_z(f)$ for any $f, g \in QM(g_1, \ldots, g_n)$. In the same manner we can also transport the notion of stability made in Definition 2.1.8 into the new situation. Although the definitions seem similar there are subtle differences with corresponding difficulties:

\footnote{Note that it is not allowed to take the inverse of an element if it is not contained in $P$.}
• The degree functions $gr_z$ might depend (heavily) on the choice of the generator $P$ of $G$.

• Let $F_n$ be the free group generated by $n$-elements. The involution of $\mathbb{C}[F_n]$ is not compatible with the involution of $\mathbb{C}[x_1, \ldots, x_n]$. Hence, one must be careful if the machinery in section 2.2 can be applied to the group algebra $\mathbb{C}[F_n]$.

Example 4.1.2. Let $p \geq 2$ be a prime and $G = \mathbb{Z}/p\mathbb{Z}$, $z = 1$ with $P = \{1\}$. For $x \in G$ the number $gr(x)$ is the smallest one such that

$$x = 1 \pm \cdots \pm 1,$$

$gr(x)$-times

Note that $gr(x) \neq gr(-x)$, i.e.

$gr(p - 1) \neq gr(1)$.

Let $G = \mathbb{Z}/p\mathbb{Z}$, $z = (1,1)$ and $P = \{1,-1\}$. In contrast to the case above we now have $gr(x) = gr(-x)$.

Definition 4.1.3. Let $f \in R[G]$. We define the length $\ell e(f)$ to be the shortest length of a sum $\sum g r_{sg}$ that satisfies $f = \sum g r_g$.

Lemma 4.1.4. The cone $\Sigma\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^2$ is closed in $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$.

Proof: Let $(f_n)_n$ be a sequence in $\Sigma\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^2$ that is convergent in $\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]$. The cone $\Sigma\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^2$ is the conic hull of the set \{a$a + b\bar{b} + (\bar{a}b + \bar{b}a)x : a, b \in \mathbb{C}$\} and by CATHEODORY we have

$$f_n = \sum_{j=1}^{m_n} a_{nj}a_{nj} + b_{nj}b_{nj} + (\bar{a}_{nj}b_{nj} + \bar{b}_{nj}a_{nj})x,$$

where the sequence $(m_n)_n$ is bounded. Since the sequence $(f_n)_n$ converges, we see that the sequences $(a_{nj}a_{nj} + b_{nj}b_{nj})_n$ and $(\bar{a}_{nj}b_{nj} + \bar{b}_{nj}a_{nj})_n$ also converge for all $j$, implying that $f \in \Sigma\mathbb{C}[\mathbb{Z}/2\mathbb{Z}]^2$.

Definition 4.1.5. Let $R[G]$ be a group ring. For an element $g = \sum g a g g \in R[G]$ we define $\text{tr}(\sum g a g g) = a_1$, where 1 is the neutral element.

Proposition 4.1.6. Let $G$ be a finite group. Then the cone $\Sigma\mathbb{C}[G]^2$ is closed.

Proof: Again take a convergent sequence $(f_n)_n$ in $\Sigma\mathbb{C}[G]^2$. We proceed as in Lemma 4.1.4. By Catheodory we can assume that the length of the sums $f_n$ is bounded. By considering $\text{tr}(f_n)$ and acknowledging that the trace maps sums of squares in $\mathbb{C}[G]$ to sums of squares in $\mathbb{C}$, we conclude that the coefficients of the sums $f_n$ converge, implying that the limit is contained in $\Sigma\mathbb{C}[G]^2$.

Theorem 4.1.7. Let $G$ be a non-trivial finite abelian group. Then $\Sigma\mathbb{C}[G]^2$ is not totally stable with respect to every $z > 0$-grading that is associated to the canonical generators.
Proof: We have
\[ G \cong \bigoplus_{j=1}^{n} \mathbb{Z}/p^j \mathbb{Z} \]
and therefore
\[ \mathbb{C}[G] \cong \bigotimes_{j=1}^{n} \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}]. \]

Step I: \( \Sigma \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}] \) is not totally stable: Without loss of generality, we can assume that \( x > 0 \). Let \( 1 \) denote the element \( 1 \cdot 0 \) in the group ring. Consider \((1 + p^{j-1})^*(1 + p^{j-1}) \) and \((-1 + p^{j-1})^*(-1 + p^{j-1}) \). Then
\[ \text{gr}_{x}((1 + p^{j-1})^*(1 + p^{j-1}) + (-1 + p^{j-1})^*(-1 + p^{j-1})) \]
is smaller than \( \text{gr}_{x}((1 + p^{j-1})^*(1 + p^{j-1}) \) resp. \( \text{gr}_{x}((-1 + p^{j-1})^*(-1 + p^{j-1}) \), since \( \text{Le}_{x}((1 + p^{j-1})^*(1 + p^{j-1}) + \text{Le}_{x}((-1 + p^{j-1})^*(-1 + p^{j-1})) = 0 \). Thus, we are done.

Step II: \( \Sigma \mathbb{C}[G]^2 \) is not totally stable: It is enough to consider the case
\[ \mathbb{C}[G] \cong \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}]. \]

Canonically embedding \( \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}] \) into \( \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}] \otimes \mathbb{C}[\mathbb{Z}/p^j \mathbb{Z}] \) shows, together with step I, that \( \mathbb{C}[G] \) cannot be totally stable with respect to the \( z \)-grading.

Corollary 4.1.8. Any finitely generated abelian group \( G \) with torsion cannot be totally stable with respect to any \( z \)-degree that is associated the canonical generators. If \( \mathbb{Z}^k \) is a direct summand of \( G \) for \( k \geq 3 \), then \( \mathbb{C}[G] \) is not even stable.

Proof: The first statement follows directly from Theorem 4.1.7. The second follows from [35, Example 4.2.4, p. 63].

Theorem 4.1.9. Let \( F_n \) be the free group of rank \( n \in \mathbb{N} \). Then the quadratic module \( \Sigma \mathbb{C}[F_n]^2 \) is stable.

Proof: That is [41, Theorem 6.1, p. 16] combined with the arguments we used in the proof of Theorem 3.5.9.

4.2 Profinite group rings and their completion

Consider the ring \( \mathbb{Z}_p = \lim_{\leftarrow n \in \mathbb{N}} \mathbb{Z}/p^n \mathbb{Z} \) of \( p \)-adic integers and the corresponding group ring \( \mathbb{C}[\mathbb{Z}_p] \). For every \( n \in \mathbb{N} \) we write
\[ \text{gr}: \mathbb{Z}/p^n \mathbb{Z} \rightarrow \mathbb{N} \]
for the \( \text{gr}_z \) with \( z = 1 \) and \( P = \{1\} \). The above mappings and the identification \( \mathbb{Z}_p \cong \mathbb{Z}[x]/(x - p) \) induce a mapping
\[ \text{gr}': \mathbb{Z}_p \rightarrow \mathbb{N} \cup \{\infty\} \]
in an obvious way. The mapping $gr'$ can be viewed as a degree mapping on $\mathbb{Z}_p$ that is induced by the degree mappings $gr: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{N}$. Note, that $gr'(f)$ can be infinite for some $f \in \mathbb{Z}_p$. In fact, $gr'(f)$ is finite if and only if the corresponding element of $f$ in $\mathbb{Z}[x]/(x - p)$ is a finite power series. Thus $gr'$ gives us a decomposition

$$\mathbb{C}[\mathbb{Z}_p] \cong \mathbb{C}[\mathbb{Z}_p]_{<\infty} \oplus \mathbb{C}[\mathbb{Z}_p]_{=\infty},$$

where $\mathbb{C}[\mathbb{Z}_p]_{<\infty} = \{ f \in \mathbb{C}[\mathbb{Z}_p] : gr'(f) < \infty \}$ and $\mathbb{C}[\mathbb{Z}_p]_{=\infty} = \{ f \in \mathbb{C}[\mathbb{Z}_p] : gr'(f) = \infty \}$.

We are interested in the features that $\mathbb{C}[\mathbb{Z}_p]$ provides with respect to the cone of sums of squares.

**Lemma 4.2.1.** Consider the group ring $B(\mathcal{H})[\mathbb{Z}_p]$ and its completion $\widehat{B(\mathcal{H})[\mathbb{Z}_p]} = \lim_{n \in \mathbb{N}} B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}]$ where the limit is taken over the category of locally convex $\mathbb{C}$-vector spaces. Then the following statements hold:

a. There is an exact sequence

$$0 \to B(\mathcal{H})[\mathbb{Z}_p] \to \widehat{B(\mathcal{H})[\mathbb{Z}_p]}.$$

b. The space $\widehat{B(\mathcal{H})[\mathbb{Z}_p]}$ is locally convex.

c. The closure $\overline{Y}$ of a subset $Y \subseteq B(\mathcal{H})[\mathbb{Z}_p]$ is given by $\overline{Y} = \lim_{n \in \mathbb{N}} \text{pr}_n(Y)$, where $\text{pr}_n : B(\mathcal{H})[\mathbb{Z}_p] \to B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}]$ is the projection.

d. The two group rings $B(\mathcal{H})[\mathbb{Z}]$ and $B(\mathcal{H})[\mathbb{Z}_p]$ are dense in $\widehat{B(\mathcal{H})[\mathbb{Z}_p]}$.

**Proof:** (a): We take the homomorphism $B(\mathcal{H})[\mathbb{Z}_p] \to \widehat{B(\mathcal{H})[\mathbb{Z}_p]}$ that factors through the canonical homomorphism $B(\mathcal{H})[\mathbb{Z}_p] \to \prod_{n \in \mathbb{N}} B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}]$.

(b): The set $\widehat{B(\mathcal{H})[\mathbb{Z}_p]}$ is a closed subset of the locally convex space $\prod_{n \in \mathbb{N}} B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}]$ and therefore again a locally convex space.

(c): That is a consequence of [10, Corollary, p. 49].

(d): We have

$$\lim_{n \in \mathbb{N}} \text{pr}_n(B(\mathcal{H})[\mathbb{Z}]) = \lim_{n \in \mathbb{N}} \text{pr}_n(B(\mathcal{H})[\mathbb{Z}_p]) = \lim_{n \in \mathbb{N}} \left( \widehat{B(\mathcal{H})[\mathbb{Z}_p]} \right) = \widehat{B(\mathcal{H})[\mathbb{Z}_p]}$$

and by (c) these equalities result in

$$\widehat{B(\mathcal{H})[\mathbb{Z}]} = \widehat{B(\mathcal{H})[\mathbb{Z}_p]} = B(\mathcal{H})[\mathbb{Z}_p],$$

finishing the proof. □

\[\text{This definition is motivated by [48, Chapter 4, pp. 453-458].}\]
Remark 4.2.2. There is the following more elaborate proof of 4.2.1 (d).

For each $n \in \mathbb{N}$ we have an exact sequence

$$0 \to \mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}] \otimes B(\mathcal{H}) \to B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}] \to 0$$

of $\mathbb{C}$-vector spaces. By the MITTAG-LEFFLER THEOREM we get an exact sequence

$$0 \to \varinjlim_{n \in \mathbb{N}} (\mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}] \otimes B(\mathcal{H})) \to \varinjlim_{n \in \mathbb{N}} B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}] \to 0.$$

Furthermore we have the sequence

$$\mathbb{C}[\mathbb{Z}] \otimes B(\mathcal{H}) \leftarrow \varinjlim_{n \in \mathbb{N}} \mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}] \otimes B(\mathcal{H}) \to \varinjlim_{n \in \mathbb{N}} (\mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}] \otimes B(\mathcal{H})), $$

which, with the sequence above, implies that the image of $\varinjlim_{n \in \mathbb{N}} \mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}] \otimes B(\mathcal{H})$ in $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$ is dense. Now the canonical inclusions

$$\mathbb{C}[\mathbb{Z}] \hookrightarrow \mathbb{C}[\mathbb{Z}_p] \hookrightarrow \varinjlim_{n \in \mathbb{N}} \mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}]$$

and the exactness of the functor $- \otimes B(\mathcal{H})$ imply that $\mathbb{C}[\mathbb{Z}_p] \otimes B(\mathcal{H}) \cong B(\mathcal{H})[\mathbb{Z}_p]$ is a dense subset of $\varinjlim_{n \in \mathbb{N}} (\mathbb{C}[\mathbb{Z}/p^n\mathbb{Z}] \otimes B(\mathcal{H})) \cong \overline{B(\mathcal{H})[\mathbb{Z}_p]}$.

Remark 4.2.3. The space $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$ has some other interesting properties that have not been tangled by Lemma 4.2.1. Consider the sequence

$$\cdots \to B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}] \to B(\mathcal{H})[\mathbb{Z}/p^{n-1}\mathbb{Z}] \to \cdots \to B(\mathcal{H}) \to 0,$$

where the mappings are induced by the canonical homomorphisms $\mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-1}\mathbb{Z}$ for $n \in \mathbb{N}$. Each of the mappings is continuous. By [32 Theorem 1, p. 374] the space $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$ is a separable FRECHET-MONTEL SPACE.

Remark 4.2.4. By Lemma 4.2.1 and Remark 4.2.2 we know that $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$ is dense in $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$. By Proposition [10 Proposition 9, p. 49] the sets

$$\{ \text{pr}_n^{-1}(U_n) : U_n \text{ open in } \overline{B(\mathcal{H})[\mathbb{Z}_p]} \}$$

form a basis of the inverse limit topology of $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$. Thus, for every $f \in \overline{B(\mathcal{H})[\mathbb{Z}_p]}$ we can find a sequence in $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$ that converges towards $f$.

Lemma 4.2.5. Let $f \in B(\mathcal{H})[\mathbb{Z}_p]$ with $f = \sum_{g} A_{g} g$ and $\ell(f) = k$. Then there is a sequence $(f_n)_n$ in $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$ such that $f_n \to f$ with $\ell(f_n) = k$ for $n \in \mathbb{N}$ big enough.

Proof: Let us view $f$ as an element of $\overline{B(\mathcal{H})[\mathbb{Z}_p]}$. We have the following commutative diagram

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Using \( \text{pr}_n \) we can take the preimage of the \( n \)-th component of \( f \) in \( B(\mathcal{H})[\mathbb{Z}] \). The preimage contains an element \( f_n \in B(\mathcal{H})[\mathbb{Z}] \) such that \( f_n = \sum_i B_i z_i \) and \( z_i < p^n \) for all \( i \). Since \( \ell e(f) = k \), there is a number \( n' \in \mathbb{N} \) such that the length of the \( n \)-th component of \( f \) is equal to \( k \) for \( n \geq n' \). Since \( z_i < p^n \), this forces \( \ell e(f_n) = k \) for \( n \geq n' \). Using Remark 4.2.4 we see that \( f_n \) converges towards \( f \) by construction.

**Remark 4.2.6.** Consider Lemma 4.2.5 and add the condition that all the elements are hermitian. I.e we get the statement:

Let \( f \in B(\mathcal{H})[\mathbb{Z}] \) be non-negative of degree \( d \geq N \). Then there is an outer analytic element \( q \in B(\mathcal{H})[\mathbb{Z}] \) of degree \( d \) such that \( f = q \cdot q \).

**Proof:** See \[19\] Theorem 1.1, p. 2.

---

\[ \cdots \rightarrow B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}] \rightarrow B(\mathcal{H})[\mathbb{Z}/p^{n-1}\mathbb{Z}] \rightarrow \cdots \]

\( \text{pr}_n \) | \( \text{pr}_{n-1} \)
---

\[ \cdots \rightarrow B(\mathcal{H})[\mathbb{Z}] \rightarrow B(\mathcal{H})[\mathbb{Z}] \rightarrow \cdots \]
Definition 4.2.8. The set of hermitian elements $B(\mathcal{H})[\mathbb{Z}_p]_{\text{her}}$ of $B(\mathcal{H})[\mathbb{Z}_p]$ is given by

$$B(\mathcal{H})[\mathbb{Z}_p]_{\text{her}} = \left( \prod_{n \in \mathbb{N}} B(\mathcal{H})[\mathbb{Z}/p^n\mathbb{Z}]_{\text{her}} \right) \cap B(\mathcal{H})[\mathbb{Z}_p].$$

Definition 4.2.9. Let $X$ be a Frechet space. A sequence $(x_n)_n$ in $X$ is called Cauchy if for every neighborhood $U$ of $0$ there is a natural number $n'$ such that $x_n - x_m \in U$ for all $n, m \geq n'$. By definition every Cauchy sequence in $X$ is convergent.

Definition 4.2.10. Let $f \in B(\mathcal{H})[\mathbb{Z}_p]_{\text{her}}$. We say that $f$ is non-negative if there is a sequence $(f_n)_{n \in \mathbb{N}}$ in $B(\mathcal{H})[\mathbb{Z}]$ that converges towards $f$ and a natural number $n'$ such that $f_n \in B(\mathcal{H})[\mathbb{Z}_p]_{\text{her}}$ is non-negative for all $n \geq n'$.

Proposition 4.2.11. The following statements hold:

a If $f \in B(\mathcal{H})[\mathbb{Z}_p]_{\text{her}}$ is non-negative, then $f \in \Sigma B(\mathcal{H})[\mathbb{Z}_p]^2$.

b Let $f \in B(\mathcal{H})[\mathbb{Z}_p]_{\text{her}}$ be non-negative and $(f_n)_n$ a sequence in $B(\mathcal{H})[\mathbb{Z}]$ that converges towards $f$. If for big $k \in \mathbb{N}$ the polynomial $f_k$ can be written as $f_k = q_k^*q_k$ such that the sequence $(q_k)_k$ is Cauchy in $B(\mathcal{H})[\mathbb{Z}_p]$, then $f = q^*q$ for some $q \in B(\mathcal{H})[\mathbb{Z}_p]$.

Proof: (a): This follows directly from Lemma 4.2.1. Since $B(\mathcal{H})[\mathbb{Z}]$ is a subset of $B(\mathcal{H})[\mathbb{Z}_p]$, we get that $\Sigma B(\mathcal{H})[\mathbb{Z}]^2$ is a subset of $\Sigma B(\mathcal{H})[\mathbb{Z}_p]^2$. Furthermore, we have

$$\lim_{n \in \mathbb{N}} \text{pr}_n (\Sigma B(\mathcal{H})[\mathbb{Z}]^2) \subseteq \Sigma B(\mathcal{H})[\mathbb{Z}_p]^2.$$ 

If an element $f \in B(\mathcal{H})[\mathbb{Z}_p]$ is non-negative, then by Remark 4.2.6 we can approximate it by a sequence $(f_n)_n$ in $B(\mathcal{H})[\mathbb{Z}]$ with $f_n \in B(\mathcal{H})[\mathbb{Z}]_{\text{her}}$ non-negative for big $n \in \mathbb{N}$. By Theorem 4.2.7 we can write $f_n$ as a square for big $n \in \mathbb{N}$. Thus, $f \in \lim_{n \in \mathbb{N}} \text{pr}_n (\Sigma B(\mathcal{H})[\mathbb{Z}]^2)$ and we are done.

(b): Follows directly from the fact that $B(\mathcal{H})[\mathbb{Z}_p]$ is a Frechet space. \(\Box\)

Remark 4.2.12. Let us take a closer look at Proposition 4.2.11 with respect to $\mathbb{R}[\mathbb{Z}_p]$ and $\mathbb{R}[\mathbb{Z}_p]$. If not otherwise stated, we will assume that $\mathbb{R}[\mathbb{Z}_p]$ is endowed with the subspace topology from $\mathbb{R}[\mathbb{Z}_p]$.

We want to know more about the sets

$$\Sigma \mathbb{R}[\mathbb{Z}_p]^2, \Sigma \mathbb{R}[\mathbb{Z}_p]^2.$$

Let us start with the identification

$$\mathbb{R}[\mathbb{Z}] \cong \mathbb{R}[S^1] \cong \mathbb{R}[x_1, x_2]/(x_1^2 + x_2^2 - 1),$$

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where $S^1$ is the unit circle. Using [49] and 4.2.7 we get the inclusions
\[
\Sigma R[Z]^2 \subseteq \Sigma C[Z]^2 \cap R[Z] \subseteq \Sigma R[Z]^2,
\]
implying that $\Sigma R[Z]^2$ equals the set of all elements in $R[Z]$ that are non-negative on the circle $S^1$. We have an exact sequence
\[
0 \to I(x_1^2 + x_2^2 - 1) \to R[x_1, x_2] \to R[S^1] \cong R[Z] \to 0
\]
and since the functor $\lim_{n \in \mathbb{N}}$, in this case, is exact, we see that $\overline{R[Z_p]}$ is a quotient of $R[[x_1, x_2]]$. By [13, Corollary 5.14, p. 68] we know that the Pythagoras number of $\overline{R[Z_p]}$ is bounded by 2. Thus we have an equality
\[
\Sigma \overline{R[Z_p]}^2 = \Sigma \overline{R[Z_p]}^2.
\]
But there is more, which we will summarize up here:

a The set $\Sigma \overline{R[Z_p]}^2 \cap R[Z]$ is strictly bigger than $\Sigma R[Z]^2$: According to [48, Corollary 4.6, p. 12] a polynomial $f \in R[x_1, x_2]$ is a sum of squares in $R[[x_1, x_2]]$ if and only if $f$ is non-negative in some neighborhood of the origin in $R^2$. We can take a polynomial $f$ that is non-negative on some small neighborhood of the origin but fails to be non-negative on $S^1$.

b The set $\{f \in R[Z_p]_{her}: f \text{ is non-negative}\}$ contains the cone $\Sigma R[Z_p]^2$: Obvious.

c The cone $\Sigma C[Z_p]^2$ is not closed in $R[Z_p]$: That is the same argument as in (a). Let $f \in R[Z_p]$ be contained in the closure of $\Sigma R[Z]^2$ but not in
\[
\{f \in R[Z_p]_{her}: f \text{ non-negative}\} \supseteq \Sigma C[Z_p]^2.
\]
By construction $f$ is contained in the closure $\overline{\Sigma R[Z_p]^2} \cap R[Z_p]$ of $\Sigma R[Z_p]^2$ but not in $\Sigma R[Z_p]^2$ itself.

d We can endow $R[Z_p]$ with the topology given by the $p$-adic topology of $Z_p$: We take the topology that is given by the coefficient wise convergence in $R[Z]$ and use the inclusion $Z \to Z_p$ to topologize $R[Z_p]$. I.e we consider $R[Z_p]$ as the topological space $\bigoplus_{z \in Z_p} (C \times Z_p)$, where $C \times Z_p$ is endowed with the product topology. By Lemma 4.2.5, $R[Z]$ becomes a dense subset of $R[Z_p]$.

Under this new topology the cone $\Sigma R[Z_p]^2$ is closed if and only if every non-negative $f \in R[Z_p]$ is a limit of a sequence $(f_n)_n \subseteq R[Z]$ such that $f_n = \sum_{i=1}^{k_n} q_{i_n}^* q_{i_n}$ for some $q_n \in R[Z]$ with the property that the sequences $\ell e(q_{i_n})_n$ and $(k_n)_n$ are bounded.
Remark 4.2.13. Let $G$ be a group and $H$ a normal subgroup of $G$. Suppose that $\Sigma[G]^2$ is stable. Unfortunately, $\Sigma[G/H]^2$ does not need to be stable. Take for example $G = \mathbb{Z} * \mathbb{Z} * \mathbb{Z}$. According to Theorem [4.1.9] we have that $\Sigma[G]^2$ is stable. Let $H$ be the kernel of the homomorphism $\mathbb{Z} * \mathbb{Z} * \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ that is induced by the canonical injections $\mathbb{Z} \hookrightarrow \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. Then $G/H \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ and by Corollary [4.1.8] the quadratic module $\Sigma[\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}]^2$ is not stable.

Proposition 4.2.14. Let $G = \ast_{i=1}^n G_i$ such that $\Sigma[G]^2$ is stable. For each $i$ let $H_i$ and $B_i$ be groups such that the sequence

$$1 \to H_i \to G_i \to B_i \to 1$$

is split exact. Let $B = \ast_{i=1}^n B_i$. Then $\Sigma[B]^2$ is stable.

Proof: For each $i = 1, \ldots, n$ let $s_i : B_i \to G_i$ be a section that we get from the split exact sequence

$$1 \to H_i \to G_i \to B_i \to 1.$$ 

The sections $s_1, \ldots, s_n$ induce a homomorphism

$$B \to G$$

resp.

$$s : \mathbb{C}[B] \to \mathbb{C}[G]$$

that replaces each $x_i \in B_i$ with $s_i(x_i)$. Let

$$\text{pr} : \mathbb{C}[G] \to \mathbb{C}[B]$$

be the homomorphism that is induced by the homomorphism $G \to B$. Take a $U$ finite dimensional $*$-subspace of $\mathbb{C}[B]$ that is spanned by $f_1, \ldots, f_n$. Let $f_i = s_i(f_i)$ for $i = 1, \ldots, n$ and $U$ the $*$-subspace of $\mathbb{C}[G]$ that is generated by the elements $f_1, \ldots, f_n$. Since $\mathbb{C}[G]$ is stable, there is a finite dimensional $*$-subspace $W$ of $\mathbb{C}[G]$ such that

$$U \cap \Sigma[G]^2 \subseteq \Sigma[W][G]^2.$$ 

Let $\overline{W} = \text{pr}(W)$.

Step I: We have $\text{pr}(U \cap \Sigma[G]^2) = \overline{U} \cap \Sigma[B]^2$: The inclusion $\subseteq$ is clear and therefore it remains to verify $\supseteq$. Let $g \in \overline{U} \cap \Sigma[B]^2$ and $g = s(g') \in \mathbb{C}[G]$ the corresponding element. By how $g$ is constructed, we see that $g \in U$. Doing the same thing with the sums of squares representation of $g'$ shows that $g' \in U \cap \Sigma[G]^2$ and therefore $g' \in \text{pr}(U \cap \Sigma[G]^2)$.

Step II: We have $U \cap \Sigma[B]^2 \subseteq \Sigma[W][B]^2$: By Step I we can lift any $\overline{g} \in \overline{U} \cap \Sigma[B]^2$ to an element in $g \in U \cap \Sigma[G]^2$ and take a representation $g \in \Sigma[W][G]^2$. Applying $\text{pr}$ leads to $\overline{g} \in \Sigma[W][B]^2$. □

\(^3\)The free product of the groups $G_1, \ldots, G_n$. 

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Remark 4.2.15. Unfortunately, Proposition 4.2.14 cannot be used to verify if $\Sigma \mathbb{C}[\prod_{i=1}^{n} \mathbb{Z}/p^k \mathbb{Z}]^2$ is stable or not, since the exact sequence

$$0 \rightarrow p^k \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/p^k \mathbb{Z} \rightarrow 0$$

does not split.

Consider $\Sigma \mathbb{C}[\prod_{i=1}^{n} \mathbb{Z}/p^k \mathbb{Z}]^2$. It is plausible that we can choose an element $(f_1, \ldots) \in \mathbb{C}[\prod_{i=1}^{n} \mathbb{Z}/p^k \mathbb{Z}]$ such that the sum of squares representation of $f_i$ becomes larger in degree or length as $i$ becomes bigger.
Bibliography


