STOCHASTIC ANALYSIS FOR LÉVY PROCESSES

CHAOS EXPANSIONS, INVARIANCES AND INFINITE-DIMENSIONAL PROCESSES

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ABSTRACT

In this thesis, new results in three fields of stochastic analysis of Lévy processes are obtained:

In a first part, invariance properties of random variables with finite variance are investigated, whose randomness originates from a real-valued Lévy process. The groups of transformation maps under consideration permute the times of the underlying Lévy process. The Wiener-Itô chaos expansion serves as an important tool for the analysis of the invariance properties due to their deterministic counterpart in the chaos kernels representing the random variable. The known structure in the chaos allows the characterization of the invariance properties by means of measurability or reduced chaos expansions for invariant subspaces of random variables. Furthermore, the notion of “local ergodicity” yields a product structure of the invariant chaos.

The second part is about Lévy processes with values in complete locally convex Suslin spaces. A useful definition and conditions for the existence of a Lévy-Itô decomposition of sample paths are elaborated. In order to prove a decomposition into independent diffusion and jump processes, a new reduction method for the small jumps of the process will be introduced. If the Lévy measure is “locally reducible”, there is a compactly embedded separable Banach space on which the small jumps are concentrated. Furthermore, the additional condition on the state space to be separably extendable ensures that every Lévy measure has this property. Banach and Fréchet spaces as well as the usual spaces of distributions and test functions share this property among many other spaces.

In the third part of the thesis, known chaos expansions for Gaussian isonormal families and Poisson random measures are adapted in order to prove chaos expansions of Wiener processes and pure-jump Lévy processes in the vector-valued case. Their combination and the results of the second part allow a short proof of a chaos expansion theorem of vector-valued Lévy processes in complete locally convex Suslin spaces.
ZUSAMMENFASSUNG

Diese Arbeit liefert neue Resultate in drei Bereichen der Stochastischen Analysis für Lévy-Prozesse:
Im dritten Teil werden bekannte Chaoszerlegungen für Gauß'sche isonormale Familien und Poisson'sche Zufallsmaße adaptiert, um sie auch für die Fälle von vektorwertigen Wiener- und reinen Sprung-Lévy-Prozessen zu erhalten. Kombiniert mit den Ergebnissen des zweiten Teils ermöglicht dies einen kurzen Beweis einer Chaoszerlegung für Lévy-Prozesse mit Werten in vollständigen lokalkonvexen Suslin-Räumen.
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Lévy processes have experienced tremendous interest during recent decades. They are a perfect model for many time-homogeneous random phenomena. The variety of applications ranges from physics (Brownian particle motion, Lévy flight, quantum field theory), meteorology and queuing theory to financial mathematics (BSDEs with Lévy noise) or insurance mathematics (modeling random losses). From a mathematical point of view, Lévy processes are natural continuous-time generalizations and limiting processes of discrete-time random walks. They are special cases of Markov processes, Feller processes and semimartingales as well as generalizations of Brownian motion, stable processes and the Poisson process. This wide range of examples illustrates the flexibility of Lévy models in many applications. Their deep connection to the class of infinitely divisible distributions leads to a vivid interplay of measure theory and the theory of stochastic processes in this area.

A core result in the theory of Lévy processes is the so-called Lévy-Itô decomposition of sample paths which enables one to filter out a Gaussian diffusion part and an independent Poisson-type jump part in every trajectory. Before K. Itô was able prove this result rigorously, P. Lévy and A. Khintchine had prepared it by their decomposition of infinitely divisible distributions in the 1930s. Even earlier, following the trace back to the roots of Lévy process, one inevitably stumbles over Bruno de Finetti, an Innsbruck born Italian mathematician. He can be credited as the founder of this area of research.

The diffusion part and the jump part are the two building blocks of Lévy processes appearing in almost every problem in this field. Their very different mathematical be-
1 Introduction

Behaviour often makes it necessary to split problems into these parts in order to obtain tractable subproblems. Mostly, this requires the knowledge of the Lévy-Itô decomposition.

One example for this playing a central role in this thesis is the chaos expansion or chaos representation of square integrable random variables which are measurable with respect to the process. In order to prove this representation, a random measure according to the Lévy-Itô decomposition has to be defined.

First attempts to generalize the chaos expansion from the known one-dimensional setting to a situation where the Lévy process takes values in a vector space, lead to the question, in which settings the Lévy-Itô decomposition was actually known. It appeared that this was the case for Banach space valued processes and for processes with values in nuclear duals of nuclear spaces. In the following, these results shall be generalized to a large class of topological vector spaces. This is the concern of the second and most extensive part of this thesis and the major step towards a chaos expansion theorem.

1.1 Outline and main results

This thesis consists of three major parts. After some preliminary remarks where well-known facts and notions are presented, Part I uses Itô’s chaos expansion as a tool for the investigation of invariance properties of Lévy random variables. The results of this part are joint work with Stefan Geiss and published in ArXiv [BG14]. Simple versions of some theorems can be found in the author’s master’s thesis [Bau11]. The main results of this part constitute the theorems 4.3.2 and 4.5.3.

Part II is dedicated to Lévy processes with values in locally convex Suslin spaces. The main result of this section is the proof of the Lévy-Itô decomposition of sample paths of such processes, Theorem 7.8.1. A main tool to obtain the decomposition is the reduction of the small-jumps-part to a Banach subspace of the state space. Two properties are essential to have: The Lévy measure must allow such a reduction and the Banach space has to be separable. These partially functional analytic questions are investigated in Section 7.3. Using zero-one laws for generalized Poisson exponents, it turns out that in spaces with a property we call separable extendability, both properties are always satisfied, cf. Subsection 7.3.3.

In Part III we recall and adapt chaos expansions for pure-jump Lévy processes and Wiener processes, each with values in locally convex Suslin spaces. The Lévy-Itô decomposition of the previous part will allow the construction of a suitable random measure in order to formulate and prove a chaos expansion theorem for these vector-valued Lévy processes, cf. Theorem 9.3.2.
Chapter 2 Preliminaries

2.1 General notation

Throughout this thesis, \((\Omega, \mathcal{F}, \mathbb{P})\) will always denote a complete probability space which is rich enough. Random variables and random elements with values in some topological space \((E, \tau)\) are mainly denoted by \(F, G\) and stochastic processes by \(X, Y, Z\), by \(X_t, Y_t, Z_t\) or by \((X_t)_{t \in T}\) in order to emphasize the parameter space \(T\). If \(F: \Omega \to E\) is a random element in \(E\), we write \(F \sim \mu\) for the fact that \(\mu = \mathbb{P}_F := \mathbb{P} \circ F^{-1}\).

For a topological space \((E, \tau)\), the Borel-\(\sigma\)-algebra generated by the open sets \(\tau\) is denoted by \(\mathcal{B}(E)\) or \(\mathcal{B}(E, \tau)\) if the choice of the topology should be emphasised. The set of measures, finite and probability measures, respectively, on \(\mathcal{B}(E)\) is denoted by \(\mathcal{M}(E)\), \(\mathcal{M}^b(E)\) and \(\mathcal{M}^1(E)\), respectively. If necessary, we indicate the underlying topology by \(\mathcal{M}(E, \tau), \mathcal{M}^b(E, \tau)\) and \(\mathcal{M}^1(E, \tau)\), respectively.

Let \(T = \mathbb{R}_+\) or \(T = [0, t_{\text{max}}]\) for \(t_{\text{max}} > 0\). The set of continuous resp. càdlàg functions \(\xi: T \to E\) is denoted by \(\mathcal{C}(T; E)\) resp. \(\mathcal{D}(T; E)\). The set of bounded and continuous functions \(f: E \to \mathbb{R}\) on a topological space \((E, \tau)\) is denoted by \(\mathcal{C}^b(E) = \mathcal{C}^b(E, \tau)\). For \(f \in \mathcal{C}^b(E, \tau)\) and \(\mu \in \mathcal{M}^b(E, \tau)\) we denote by

\[
\mu(f) := \int_E f \, d\mu.
\]
If \( E \) is a topological vector space, \( E' \) denotes its topological dual space, i.e. the space of linear continuous functionals on \( E \). The evaluation map \( \langle \cdot, \cdot \rangle : E \times E' \to \mathbb{R} \) is given by \( (x, a) \mapsto \langle x, a \rangle := a(x) \).

Further notation: \( \mathbb{N} = \{1, 2, \ldots \} \) are the natural numbers, \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \); The set \( B_1 := \{x \in E : d(x, 0) \leq 1\} \) is the closed unit ball in a metric space \((E, d)\). Lebesgue measure and its various restrictions to subintervals of \( \mathbb{R} \) are denoted by \( \lambda \).

### 2.2 Lévy processes

For notational convenience and later use we introduce well-known notions, notations and results for Lévy processes. The monographs of Sato [Sat99] and Applebaum [App09] serve as references. Let \( T = [0, \infty) \) or \( T = [0, t_{\text{max}}) \), \( t_{\text{max}} > 0 \).

**Definition 2.2.1 (Infinite divisibility).** A probability measure \( \mu \in \mathcal{M}^1(\mathbb{R}^d) \) is called **infinitely divisible** if for every \( n \in \mathbb{N} \) there exists a measure \( \mu_n \in \mathcal{M}^1(\mathbb{R}^d) \) such that

\[
\mu = \mu_n * \ldots * \mu_n.
\]

The measure \( \mu_n \) is sometimes called an \( n \)-th root of \( \mu \). The set of infinitely divisible measures on \( \mathbb{R}^d \) is denoted by \( \mathcal{I}(\mathbb{R}^d) \).

It should be noted that \( \mu_n \) is unique, cf. [Sat99, Lemma 7.6]. Furthermore, it is possible to define the infinitely divisible measure \( \mu^t \) for \( t \in [0, \infty) \) in a consistent way, cf. [Sat99, Lemma 7.9]. The following representation is an essential result on the structure of infinitely divisible measures.

**Theorem 2.2.2 (Lévy-Khintchine decomposition).** If \( \mu \in \mathcal{I}(\mathbb{R}^d) \) there exists \( \gamma \in \mathbb{R}^d \), a nonnegative definite symmetric matrix \( A \in \mathbb{R}^{d \times d} \) and a \( \sigma \)-finite measure \( \nu \in \mathcal{M}(\mathbb{R}^d) \) such that the Fourier transform \( \hat{\mu} \) has the form

\[
\hat{\mu}(z) = \exp \left( i\langle \gamma, z \rangle - \frac{1}{2} \langle Az, z \rangle + \int_{\mathbb{R}^d} \left( e^{i(x, z)} - 1 - i(x, z) \mathbb{1}_{B_1}(x) \right) d\nu(x) \right), \tag{2.2.1}
\]

\( z \in \mathbb{R}^d \). The elements \( \gamma, A \) and \( \nu \) are unique in this representation and \( (\gamma, A, \nu) \) is therefore called **characteristics** or **characteristic triplet** of \( \mu \).

A proof of this result can be found e.g. in [Sat99, Section 8]. The measure \( \nu \) is called **Lévy measure** of \( \mu \) and satisfies \( \nu(\{0\}) = 0 \) and the integrability condition

\[
\int_{\mathbb{R}^d} (|x|^2 \wedge 1) d\nu(x) < \infty.
\]
2.2 Lévy processes

**Definition 2.2.3** (Convolution semigroups). A family of measures \((\mu_t)_{t \in T} \subseteq \mathcal{M}^1(\mathbb{R}^d)\) is a *convolution semigroup* if

(C1) \(\mu_s \ast \mu_t = \mu_{s+t}\) for all \(s, t, s+t \in T\) and

(C2) \(\mu_0 = \delta_0\),

and it is *weakly continuous* if

(C3) \(\mu_t \to \delta_0\) weakly for \(t \searrow 0\), i.e., \(\mu_t(f) \to \delta_0(f), t \searrow 0\), for all \(f \in \mathcal{C}^0(\mathbb{R}^d)\).

For \(\mu \in \mathcal{I}(\mathbb{R}^d)\), the family \((\mu_t^t)_{t \in [0, \infty)}\) is a weakly continuous convolution semigroup. This means that \(\mu\) is *embeddable* in a convolution semigroup which is not always true for more general state spaces than \(\mathbb{R}^d\), cf. [Sie74].

**Definition 2.2.4** (\(\mathbb{R}^d\)-valued Lévy process). A stochastic process \(X: \Omega \times T \to \mathbb{R}^d\) is a *Lévy process in law* if the following is satisfied:

(L1) \(X_0 = 0\) a.s.,

(L2) \(X_t - X_s \sim X_{t-s}\) for \(s < t\) and \(s, t \in T\) (stationary increments),

(L3) \(X_{t_0}, X_{t_1}, \ldots, X_{t_n}\) are independent for all \(t_0 < \ldots < t_n, t_1, \ldots, t_n \in T\) (independent increments),

(L4) \(\lim_{s \searrow t} \mathbb{P}(|X_t - X_s| > \varepsilon) = 0\) for all \(\varepsilon > 0\) and \(t \in T\) (stochastic continuity).

\(X\) is a *Lévy process* if additionally,

(L5) \(X\) has càdlàg paths almost surely.

It follows directly from the definition of Lévy processes in law that for every \(t \in T\) the distribution \(\mu_t := \mathbb{P}_{X_t}\) is infinitely divisible. Furthermore, \((\mu_t)_{t \in T}\) is a weakly continuous convolution semigroup. The latter follows from stochastic continuity of \(X\).

Indeed, instead of (L4) one could equivalently pose the condition

(L4') The family of distributions \(\mu_t = \mathbb{P}_{X_t}\) is weakly continuous in \(t\).

**Definition 2.2.5** (Random measures). Let \((S, \mathcal{S})\) be a measurable space. A map \(N: S \times \Omega \to \mathbb{R}\) is a *random measure* if

1. \(N(\cdot, \omega)\) is an additive set function on \(\mathcal{S}\) with \(N(\emptyset, \omega) = 0\) for almost all \(\omega \in \Omega\),

2. \(N(A, \cdot)\) is a random variable for all \(A \in \mathcal{S}\).

\(N\) is *independently scattered* if for all \(A_1, \ldots, A_n \in \mathcal{S}, n \in \mathbb{N}\), the random variables \(N(A_1), \ldots, N(A_n)\) are independent. A \(\sigma\)-finite measure \(m\) on \((S, \mathcal{S})\) is called *intensity measure* of \(N\) if \(\mathbb{E}[N(A)N(B)] = m(A \cap B)\) for all \(A, B \in \mathcal{S}\) with \(m(A), m(B) < \infty\).
2 Preliminaries

**Definition 2.2.6** (Poisson random measure). An independently scattered random measure $N$ on a $\sigma$-finite measure space $(S, \mathcal{S}, m)$ is called a **Poisson random measure** with intensity measure $m$ if $N(A)$ is Poisson distributed for all $A \in \mathcal{S}$ with $m(A) < \infty$ and $\mathbb{E}N(A) = \mathbb{E}N(A)^2 = m(A)$.

As a Lévy process has a.s. càdlàg paths, the **jump process** $\Delta X_t := X_t - X_{t-}$ with $X_{t-} := \lim_{s \uparrow t} X_s$ is well-defined on a set $\Omega_0 \in \mathcal{F}$ of measure one. If $\omega \in \Omega_0$ set $\Delta X_t(\omega) := 0$ for all $t$. For $A \in \mathcal{B}(T \times \mathbb{R}^d)$, define

$$N(A) := \# \{ t \in T : (t, \Delta X_t) \in A \}.$$ 

$N$ is called the **Poisson random measure associated to $X$** with values in $\{\infty, 0, 1, 2, \ldots\}$ and it is indeed a Poisson random measure on $\mathcal{B}(T \times \mathbb{R}^d)$ with intensity measure $\lambda \otimes \nu$, [Sat99, Theorem 19.2 (i)]. Here, $N(\cdot, \omega)$ is not only additive but even a measure on $\mathcal{B}(T \times \mathbb{R}^d)$ for fixed $\omega \in \Omega$. Also the **compensated Poisson random measure** $\tilde{N} := N - \lambda \otimes \nu$ defined on sets $A \in \mathcal{B}(T \times \mathbb{E})$ with $\lambda \otimes \nu(A) < \infty$ is an independently scattered random measure with intensity measure $\lambda \otimes \nu$.

The intimate relation between infinitely divisible measures and Lévy processes is illustrated by the main theorem on Lévy processes, the **Lévy-Itô decomposition of sample paths**, which consists of four ingredients with counterparts in the Lévy-Khintchine decomposition of $\mathbb{F}_{X_1} \in \mathcal{I}(\mathbb{R}^d)$.

**Theorem 2.2.7** (Lévy-Itô decomposition, $\mathbb{R}^d$). Let $X = (X_t)_{t \in T}$ be a Lévy process. Then, there exists $\gamma \in \mathbb{R}^d$, a Brownian motion $(B_t)_{t \in T}$ with covariance matrix $A \in \mathbb{R}^{d \times d}$, an independently scattered Poisson random measure $N$ with intensity measure $\lambda \otimes \nu$ and a set $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$X_t(\omega) = \gamma t + B_t(\omega) + \int_{[0,t] \times B_t} x \, dN(s,x)(\omega) + \int_{[0,t] \times B_t} x \, d\tilde{N}(s,x)(\omega). \quad (2.2.2)$$

for all $t \in T$ and $\omega \in \Omega_0$. Furthermore, the summands are independent processes.

Different proofs of this well-known result can be found in [App09, Section 2.4] and [Sat99, Section 20].
2.3 Chaos expansions

Chaos expansions are series representations of square-integrable random variables in terms of deterministic functions. The main tool is the construction of so-called multiple integrals with respect to suitable random measures. This goes back to the original papers of Itô [Itô51] and [Itô56]. This approach is also possible for so-called white noise isonormal Gaussian families, cf. [Nua95], and very general Poisson point processes [LP11]. But also other approaches have been investigated including orthogonal polynomials [Nua95], [SU08] or Teugels martingales [NS00]. A detailed discussion on chaos expansions can be found in Chapter 8.

In this section, we consider a real-valued Lévy process over the time parameter space $T = [0, \infty)$ or $T = [0, t_{\max}]$ with characteristic triplet $(\gamma, \sigma, \nu)$, where $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and $\nu$ is a Lévy measure on $\mathbb{R}$. Define the $\sigma$-finite measures $d\mu(x) := \sigma^2 d\delta_0(x) + x^2 d\nu(x)$, $d\nu(t, x) := d(\lambda \otimes \mu)(t, x)$ on $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(T \times \mathbb{R})$, respectively.

**Definition 2.3.1 (Lévy random measure).** The Lévy random measure associated to the Lévy process $X$ is defined by

$$M(A) := \sigma \left( \int_{A \cap (T \times \{0\})} dB_t \right) + \lim_{N \to \infty} \int_{A \cap (T \times \{ \frac{1}{N} < |x| < N \})} x d\tilde{N}(t, x),$$

(2.3.1)

for $A \in \mathcal{B}(T \times \mathbb{R})$ with $m(A) < \infty$. Here, $B$ is the Brownian motion part of $X$ and the limit in the last term is taken in $L^2$. Note that $M(A) \in L^2(\mathcal{F}^X) := L^2(\Omega, \mathcal{F}^X, \mathbb{P})$, where $\mathcal{F}^X$ is the completed $\sigma$-algebra generated by increments of the Lévy process $X$.

For $n \in \mathbb{N}$ let

$$L^2_n := L^2((T \times \mathbb{R})^n, \mathcal{B}((T \times \mathbb{R})^n), m^\otimes n)$$

and its subset of simple functions

$$C^2_n := \left\{ f \in L^2_n : f = 1_{A_1} \cdots 1_{A_n}, A_1, \ldots, A_n \in \mathcal{B}(T \times \mathbb{R}) \right\}.$$

Elements of $L^2_n$ are called kernels or $(n$-th) chaos kernels. For $f_n \in C^2_n$ with representation $f_n = 1_{A_1} \cdots 1_{A_n}$ one defines the multiple integral with respect to $M$ by

$$I_n(f_n) := M(A_1) \cdots M(A_n) \in L^2(\mathcal{F}^X).$$
2 Preliminaries

By linearity and continuity, $I_n$ extends to a linear operator $I_n : L_n^2 \to L^2(F^X)$. For $n = 0$ set $L_0^2 := \mathbb{R}$ and $I_0(f_0) = f_0 \in \mathbb{R}$. For $n \neq m$ the integrals $I_n(f_n)$ and $I_m(f_m)$ are orthogonal for any kernels $f_n \in L_n^2$ and $f_m \in L_m^2$. A kernel $f_n$ is called symmetric provided that

$$f((t_1, x_1), \ldots, (t_n, x_n)) = f((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)}))$$

for all $(t_1, x_1), \ldots, (t_n, x_n) \in T \times \mathbb{R}$ and $\pi \in S_n$, where $S_n$ is the set of all permutations acting on $\{1, \ldots, n\}$. The symmetrization of an $f_n \in L_n^2$ is given by

$$\tilde{f}_n((t_1, x_1), \ldots, (t_n, x_n)) := \frac{1}{n!} \sum_{\pi \in S_n} f((t_{\pi(1)}, x_{\pi(1)}), \ldots, (t_{\pi(n)}, x_{\pi(n)})).$$

The most notable properties of the symmetrization in this context are invariance of the multiple integral under symmetrisation, i.e. $I_n(f_n) = I_n(\tilde{f}_n)$ a.s., and the isometry property for symmetric functions $\|I_n(\tilde{f}_n)\|_{L^2(F^X)} = \sqrt{n!} \|f_n\|_{L_n^2}$. Due to these important facts we introduce the space $\tilde{L}_n^2$ of equivalence classes of symmetric functions in $L_n^2$. The following theorem is the chaos expansion result of Itô and establishes the isometric correspondence between sequences of symmetric chaos kernels and $L^2$-random variables.

**Theorem 2.3.2** (Chaos expansion, [Itô56]). For $F \in L^2(F^X)$ there exist unique kernels $f_n \in \tilde{L}_n^2$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{in} \quad L^2(F^X).$$

Let $H_n := I_n(L_n^2) \subseteq L^2(F^X)$. The map

$$\mathcal{J} : \bigoplus_{n=0}^{\infty} \tilde{L}_n^2 \longrightarrow L^2(F^X) \cong \bigoplus_{n=0}^{\infty} H_n,$$

$$(f_n)_{n=0}^{\infty} \mapsto \sum_{n=0}^{\infty} I_n(f_n),$$

defines an isometric bijection, where $\bigoplus_{n=0}^{\infty} H_n$ is the $\ell^2$-product and $\bigoplus_{n=0}^{\infty} \tilde{L}_n^2$ is equipped with the norm

$$\|(f_0, f_1, \ldots)\| := \left( \sum_{n=0}^{\infty} n! \|f_n\|^2 \right)^{\frac{1}{2}}.$$
I

Invariance Principles of Lévy Random Variables
This part (chapters 3 and 4) contains original passages from the paper
Permutation Invariant Functionals of Lévy Processes [BG14] which is
joint work with my supervisor Stefan Geiss and has been submitted
to TAMS.

Itô’s chaos expansion as presented in Theorem 2.3.2, provides a useful tool to de-
scribe and investigate properties of random variables using their representation by the
deterministic chaos kernels.
For example, measurability with respect to \( \mathcal{F}_t^X \), the completion of \( \sigma(X_s : s \in [0,t]) \),
can be checked by the support of the chaos kernels. Malliavin differentiability or
fractional Malliavin differentiability obtained by real interpolation can be formulated
by moment conditions on the kernels [GGL13].

Already Itô himself used this chaos expansions in order to determine the spectral type
of time-shift induced operators on \( L^2(\Omega, \mathcal{F}^X, P) \) in his initial paper [Itô56]. Almost
forgotten for forty years, Itô’s result has been rediscovered in the 1990s and has been
experiencing great interest, mainly in connection with Malliavin calculus or covariance
relations of Lévy processes or general Poisson processes.

For example, it was used to investigate quantitative properties of stochastic pro-
cesses in continuous time or to prove covariance relations and inequalities, like the
Poincaré inequality, for Lévy processes and/or general Poisson processes, cf. [BDL⁺03,
3 Introduction

SUV07a, SUV07b, SU08, PSTU10, LP11, GL11, GGL13, BL14, GS13] and most recently, [EV15].

A general obstacle for the application of the chaos representation is the fact that the chaos kernels depend on an increasing number of coordinates. As a result, their structure gets involved and computations become difficult or sometimes impossible. Only in certain situations it is known how to represent the kernel functions explicitly. Using iterative difference operators or Malliavin derivatives, kernel representations are obtained in [GL11] and [LP11]. This requires very smooth random variables in the Malliavin sense, though. A different kind of restriction is posed in [YSO10], where the random variables are assumed to be powers of increments of the underlying Lévy process. Another computation of chaos kernels in a very specific situation can be found in [ÜZ92].

The approach which should be presented here, takes a different path: Studying natural invariance properties with respect to time permutations of the underlying process we are able to characterize an entire class of simpler chaos expansions for these invariant random variables. This means, we find $L^2$-spaces which serve as kernel spaces for certain closed subspaces $H \subseteq L^2(F^X)$.

Types of Invariances

In order to illustrate the variety of possible invariances, we have a look at three elementary examples. Let us define the $L^2$-random variables

$$F_1 := \Phi_1 \left( \int_{(0,1/2]} \varphi_t \, dX_t, \int_{(1/2,1]} \varphi_{t-1/2} \, dX_t \right),$$

$$F_2 := \Phi_2([X]_{1/2}, [X]_1 - [X]_{1/2}),$$

$$F_3 := \int_0^1 \int_0^t h(t-s) \, dW_s \, dW_t$$

where $\varphi: [0,1/2] \to \mathbb{R}$ is continuous, $\Phi_1: \mathbb{R}^2 \to \mathbb{R}$ symmetric, bounded and measurable, $\Phi_2: \mathbb{R}^2 \to \mathbb{R}$ bounded and measurable, but not necessarily symmetric, $h: [0,1] \to \mathbb{R}$ bounded and measurable with the symmetry $h(1/2-r) = h(1/2+r)$ for $r \in [0,1/2]$, and $W$ is the normalized Brownian part of $X$. The time variables of the corresponding symmetric kernels in the second chaos have the symmetries indicated in Figure 3.0.1. In fact, there are two interacting symmetry groups: the general symmetry in $(t_1,x_1)$ and $(t_2,x_2)$, and the symmetries coming from $\Phi_1$, the bracket process $([X]_t)_{t \in [0,1]}$ and from $h$.

**Example 3.0.1** (Example $F_1$). Example $F_1$ is invariant with respect to an interchange of the Lévy process on $(0,1/2]$ with the process on $(1/2,1]$ in the sense that
(\(X_t\))_{t \in [0,1]} is replaced by

\[
Y_t := \begin{cases} 
X_{t+1/2} - X_{1/2} & t \in [0, 1/2], \\
(X_1 - X_{1/2}) + X_{t-1/2} & t \in (1/2, 1].
\end{cases}
\]

Freezing the state variables \((x_1, x_2)\) of the kernel, this leads to a symmetry in the time variables \((t_1, t_2)\), where the areas \(A'\) resp. \(B'\) are copies of \(A\) resp. \(B\) obtained by a shift. The remaining parts are determined by the symmetry in \((t_1, x_1)\) and \((t_2, x_2)\).

**Example 3.0.2** (Example \(F_2\)). Similarly as described above, the Lévy process can be exchanged on all intervals within \((0, \frac{1}{2}]\) resp. \((\frac{1}{2}, 1]\). Later we show that this immediately results in the structure

\[
f_2((t_1, x_1), (t_2, x_2)) = \mathbb{I}_C(t_1, t_2)g_C(x_1, x_2) + \mathbb{I}_E(t_1, t_2)g_E(x_1, x_2) + \\
\mathbb{I}_D(t_1, t_2)g_D(x_1, x_2) + \mathbb{I}_{DT}(t_1, t_2)g_D(x_2, x_1),
\]

where the functions \(g_C\) and \(g_E\) appearing in the diagonal terms are already symmetric.

**Example 3.0.3** (Example \(F_3\)). As we only consider the Brownian motion, there is no dependence of the kernel on the state variables \(x_1\) and \(x_2\). Directly, from the symmetries of \(h\), one checks that the kernel is constant in time along the lines indicated in Figure 3.0.1, whereas on lines with the same color the kernel takes the same values.

**Abstract chaos properties**

Symmetries as described above are the basis to restrict the chaos expansion to a subspace \(H\). Let us list some desired abstract properties of the restricted chaos expansion.

(S) **Stability**: Given random variables \(F_1, \ldots, F_N \in H\) and a suitable bounded random functional \(f : \Omega \times \mathbb{R}^N \to \mathbb{R}\), such that \(f(\cdot, x) \in H\) for all \(x \in \mathbb{R}^N\), we
would like to guarantee that $f(F_1, \ldots, F_N) \in H$.

(C) **Consistency:** We consider three different stages of compatibility of $H$ with the original chaos decomposition.

(C1) Are there closed linear subspaces $H_n \subseteq H_n$ such that

$$H = L^2 - \bigoplus_{n=0}^{\infty} H_n?$$

(C2) Can the subspaces $H$ and $H_n$ be obtained by measurability, i.e. are there $\sigma$-algebras $A$ and $A_n$ such that

$$H = L^2(\Omega, A, P) \text{ and } H_n = I_n \left( L^2((0, 1] \times \mathbb{R})^n, A_n, m \otimes^n \right)?$$

(C3) Can one realize $A_n = A_1^{\otimes n}$?

(G) **Generating property of $H_1$:** Does one have that

$$A = \sigma(F \in H_1) \vee \{ A \in \mathcal{F}^X : P(A) = 0 \}.$$  

Before we proceed let us comment on the above conditions:

- Roughly speaking, property (C2) is stronger than the stability property (S): If the map $\omega \mapsto f(\omega, F_1(\omega), \ldots, F_N(\omega))$ can be defined in a reasonable way, then the measurability will transfer automatically to the composition and implies $f(F_1, \ldots, F_N) \in H$ by (C2).

- The stability (S) excludes certain choices of $H$ such as $H = H_n$ for some $n \geq 1$.

- The generating property (G) holds for Itô’s chaos expansion as introduced above, and might be approached by orthogonal polynomials associated to certain Lévy processes (cf. [NS00, SU08, Pri09]) in order to obtain other cases. For example, it holds for the Hermite expansion of the Gaussian space $(\mathbb{R}^n, B(\mathbb{R}^n), \gamma_n)$ with $\gamma_n$ being the standard Gaussian measure on $\mathbb{R}^n$, and for functionals $f(N_t)$, where $(N_t)_{t \in [0,1]}$ is a standard Poisson process by exploiting Charlier polynomials [Pri09, Chapter 6].

- In general, condition (C2) does not imply (C3) nor (G): take for $H'$ the space of random variables that are invariant with respect to all dyadic permutations of the underlying Lévy process, and for $H''$ the space of random variables being invariant with respect to all dyadic periodic shifts of the underlying Lévy process. We have $H' \subseteq H'$ and in Section 4.6.1 we provide an example that $H' \not\subseteq H''$. 

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Because $g \in L^2((0, 1])$ is a.s. constant if and only if $g$ is a.s. invariant with respect to all periodic dyadic shifts, in both cases the first chaos coincides and equals

$$H_1' = H_1'' = \{I_1(1_{(0,1)}|1_1) : 1_{(0,1)}|1_1 \in L^2((0, 1] \times \mathbb{R}, m)\},$$

where $(1_{(0,1)}|1_1)(t_1, x_1) = g_1(x_1)$. Using Theorem 4.5.3(2) for $L = 1$ and $E_1 = (0, 1]$ gives properties (C3) and (G) for $H'$, so that (C3) and (G) cannot hold for $H''$ as $H' \subsetneq H''$. This also means, although $g \in L^2((0, 1])$ is a.s. constant whenever $g$ is invariant with respect to all shifts, this phenomenon does not transfer to Itô’s chaos representation.

We will give a complete answer to the questions (S), (C1) and (C2) in Theorem 4.3.2. To handle the properties (C3) and (G), we introduce the concept of a **locally ergodic set** in Definition 4.4.1 and obtain a corresponding result in Theorem 4.5.3.
Throughout this chapter let $X = (X_t)_{t \in [0,1]}$ be a Lévy process with Gaussian part $\sigma \cdot B_t$, drift $\gamma$ and Lévy-Itô decomposition as defined in Section 2.2 and chaos expansion as in Section 2.3.

### 4.1 Dyadic permutations and Lévy processes

In this section we investigate measure preserving transformations on $L^2(\mathcal{F}^X)$ and on the chaos decomposition induced by dyadic measure preserving maps $g: (0,1] \to (0,1]$. The final commutative diagram will be

$$
\begin{align*}
L^2(\mathcal{F}^X) & \xrightarrow{T_g} L^2(\mathcal{F}^X) \\
\bigoplus_{n=0}^{\infty} L^2_n & \xrightarrow{S_{g^{-1}}} \bigoplus_{n=0}^{\infty} \tilde{L}^2_n
\end{align*}
$$

and is verified in Theorem 4.1.9 below. This diagram transfers Lemma 1 of [Itô56], where shift operations are considered, to our setting. The diagram is based on the
fact that by the definition of Lévy processes, their increments are exchangeable. Later we investigate how this exchangeability transfers to certain functionals defined on the process \( X \) or more generally, to \( L^2(\mathcal{F}^X) \)-random variables. In order to shorten the presentation, given \( 0 \leq a < b \leq 1 \) and \( I := (a, b] \), we let \( X_I := X_b - X_a \). The dyadic intervals we denote by

\[
I^d_k := \left( \frac{k-1}{2^d}, \frac{k}{2^d} \right] \quad \text{for } d \geq 0 \text{ and } k \in \{1, \ldots, 2^d\}.
\]

### 4.1.1 Construction of \( T_g \)

For an integer \( d \geq 0 \) we let

\[
\mathcal{H}^{X,d} := \left\{ F \in L^2(\mathcal{F}^X) : F = f(X_{I^d_k}, \ldots, X_{I^d_{2^d}}), \ f \in C^b(\mathbb{R}^{2^d}) \right\}
\]

and

\[
\mathcal{H}^X := \bigcup_{d \geq 0} \mathcal{H}^{X,d}.
\]

All spaces \( \mathcal{H}^{X,0} \subseteq \mathcal{H}^{X,1} \subseteq \cdots \subseteq \mathcal{H}^X \) are linear subspaces of \( L^2(\mathcal{F}^X) \).

**Lemma 4.1.1.** \( \mathcal{H}^X \) is dense in \( L^2(\mathcal{F}^X) \).

**Proof.** It is known that

\[
\left\{ f(X_{(t_0,t_1)}, \ldots, X_{(t_{N-1},t_N)}) : 0 \leq t_0 < \ldots < t_N \leq 1, \ f \in C^b(\mathbb{R}^N), N \in \mathbb{N} \right\}
\]

is dense in \( L^2(\mathcal{F}^X) \), cf. for example [Itô56]. The right-continuity of \((X_t)_{t \in [0,1]}\) yields our assertion. \( \square \)

**Definition 4.1.2.** (1) For \( d \geq 0 \) and \( \pi \in S_{2^d} \) we define \( g_\pi : (0,1] \to (0,1] \) by shifting \( I^d_k \) onto \( I^d_{\pi(k)} \), i.e.,

\[
g_\pi(t) := \frac{\pi(k)}{2^d} - \left( \frac{k}{2^d} - t \right) \quad \text{if } t \in \left( \frac{k-1}{2^d}, \frac{k}{2^d} \right].
\]

(2) We let \( M_{\text{dyad}} := \{ g_\pi : (0,1] \to (0,1] : \pi \in S_{2^d}, \ d \geq 0 \} \).

(3) We say that \( g \in M_{\text{dyad}} \) is represented by \( \pi \in S_{2^d} \) for some \( d \geq 0 \) if \( g = g_\pi \).

(4) For \( g \in M_{\text{dyad}} \) we let \( \deg(g) := \min d \), where the minimum is taken over all \( d \geq 0 \) such that \( g \) can be represented by some \( \pi \in S_{2^d} \).

Note that for \( d \geq \deg(g) \) the map \( g \) can always be represented by some \( \pi \in S_{2^d} \) and that all \( g \in M_{\text{dyad}} \) preserve the Lebesgue measure.
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**Definition 4.1.3.** For \( g \in M_{\text{dyad}}^d \), \( d \geq \deg(g) \), and \( \pi \in S_{2d} \) representing \( g \), we define the operator \( T_g : \mathcal{H}^{X,d} \rightarrow \mathcal{H}^{X,d} \) by

\[
T_g f(X_{I_{d}}, \ldots, X_{I_{2d}}) := f(X_{I_{\pi(1)}}, \ldots, X_{I_{\pi(2d)}}).
\]

**Lemma 4.1.4.**

1. For \( d \geq \deg(g) \) the operator \( T_g : \mathcal{H}^{X,d} \rightarrow \mathcal{H}^{X,d} \) is well defined.
2. For \( e \geq d \geq \deg(g) \) the operators \( T_g : \mathcal{H}^{X,d} \rightarrow \mathcal{H}^{X,d} \) and \( T_g : \mathcal{H}^{X,e} \rightarrow \mathcal{H}^{X,e} \) are consistent.
3. For \( F \in \mathcal{H}^{X,d} \) with \( d \geq \deg(g) \) the random variables \( F \) and \( T_g F \) have the same distribution. In particular, \( T_g \) is a linear isometry in \( L^2(F^X) \).

**Proof.** (1) Assume that

\[
f_1(X_{I_{d}}, \ldots, X_{I_{2d}}) = f_2(X_{I_{d}}, \ldots, X_{I_{2d}}) \quad \text{a.s.}
\]

Because of the exchangeability of the increments of the Lévy process, the permuted vector of increments has the same distribution as the original vector. This implies

\[
f_1(X_{I_{\pi(1)}}, \ldots, X_{I_{\pi(2d)}}) = f_2(X_{I_{\pi(1)}}, \ldots, X_{I_{\pi(2d)}}) \quad \text{a.s.}
\]

and the equivalence classes coincide.

Assertion (2) follows from the definition and assertion (3) follows by the same distributional argument as in (1).

Because of Lemma 4.1.4 we can extend \( T_g \) to an \( L^2 \)-isometry \( T_g : \mathcal{H}^X \rightarrow \mathcal{H}^X \), and by Lemma 4.1.1, we obtain an isometry

\[
T_g : L^2(F^X) \rightarrow L^2(F^X).
\]

The operator \( T_g \) acts on the jump-part of \( X \) as follows:

**Lemma 4.1.5.** Let \( g \in M_{\text{dyad}}^d \), \( N \) be the Poisson random measure associated to \( X \), \( I = (a, b] \) with \( 0 \leq a < b \leq 1 \) being dyadic, and \( E = (c, d) \) with \(-\infty < c < d < \infty \) and \( 0 \notin E \). Then,

\[
T_g \int_{I \times E} x \, dN(s, x) = \int_{g(I) \times E} x \, dN(s, x) \quad \text{a.s.}
\]

**Proof.** The proof follows an idea of [GL11]. We show that for \( L \in \mathbb{N} \) and the truncation
Proof. As in the proof of Lemma 4.1.5 it is enough to prove that

\[ T_g \psi_L \left( \int_{I \times E} x \, dN(s, x) \right) = \psi_L \left( \int_{g(I) \times E} x \, dN(s, x) \right) \text{ a.s.} \]

Then the assertion follows from the fact that \( \psi_L(F) \) converges in \( L^2(\mathcal{F}^X) \) to \( F \) whenever \( F \in L^2(\mathcal{F}^X) \). For \( l \in \mathbb{N} \) with \( 2/l < d - c \) we define a continuous function \( h_l \) such that \( h_l(x) = x \) on \( [c + (1/l), d - (1/l)] \), \( h_l(x) = 0 \) if \( x \notin [c, d] \) and on the remaining parts we take the linear interpolation. By construction, \( \lim_{l \to \infty} h_l(x) = x1_E(x) \) and \( |h_l(x)| \leq |x1_E(x) \). By definition,

\[
T_g \psi_L \left( \sum_{k=1}^{2^n} h_l \left( X_{g(I)}^T \right) \right) = \psi_L \left( \sum_{k=1}^{2^n} h_l \left( X_{g(I)}^T \right) \right) \quad \text{a.s.,}
\]

where we assume that \( n \geq \deg(g) \lor n_0 \), with \( n_0 \geq 0 \) chosen such that \( a \) and \( b \) belong to the dyadic grid with mesh size \( 2^{-n_0} \). Using the fact that for a fixed càdlàg path \( t \to \xi_t = X_t(\omega) \) and for any \( \varepsilon > 0 \) one finds a partition \( 0 = t_0 < \cdots < t_N = 1 \) such that for all \( t_{i-1} \leq s < t < t_i \) one has that \( |\xi_t - \xi_s| \leq \varepsilon \) (see [Bil99, Lemma 1, Chapter 3]), one concludes by \( n \to \infty \) with dominated convergence that

\[
T_g \psi_L \left( \sum_{t \in (a,b]} h_l(\Delta X_t) \right) = \psi_L \left( \sum_{t \in (a,b]} h_l(\Delta X_{g(t)}) \right) \quad \text{a.s.}
\]

Letting \( l \to \infty \) and using again dominated convergence finally yields

\[
T_g \psi_L \left( \sum_{t \in (a,b]} \Delta X_t 1_E(\Delta X_t) \right) = \psi_L \left( \sum_{t \in (a,b]} \Delta X_{g(t)} 1_E(\Delta X_{g(t)}) \right) \quad \text{a.s.} \quad \Box
\]

**Lemma 4.1.6.** Let \( g \in \mathbb{M}^{\text{dyad}}, F_1, \ldots, F_n \in L^2(\mathcal{F}^X) \) and \( f: \mathbb{R}^n \to \mathbb{R} \) be continuous such that \( f(F_1, \ldots, F_n) \in L^2(\mathcal{F}^X) \). Then

\[
T_g f(F_1, \ldots, F_n) = f(T_g F_1, \ldots, T_g F_n) \quad \text{a.s.}
\]

**Proof.** As in the proof of Lemma 4.1.5 it is enough to prove that

\[
T_g \psi_L(f(F_1, \ldots, F_n)) = \psi_L(f(T_g F_1, \ldots, T_g F_n)) \quad \text{a.s.}
\]

so that we can assume that \( f \in C^b(\mathbb{R}^n) \). By Lemma 4.1.1, we find \( H^X \ni F_{i,k} \to F_i \) in \( L^2(\mathcal{F}^X) \) as \( k \to \infty \). By a diagonal argument, we find a sub-sequence \( (k_l)_{l=1}^\infty \) such
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that, for \( l \to \infty \), \( F_{i,k_l} \to F_i \) a.s. and \( T_g F_{i,k_l} \to T_g F_i \) a.s. for \( i = 1, \ldots, n \). Therefore, as \( l \to \infty \),

\[
f(F_{1,k_l}, \ldots, F_{n,k_l}) \to f(F_1, \ldots, F_n)
\]

and

\[
f(T_g F_{1,k_l}, \ldots, T_g F_{n,k_l}) \to f(T_g F_1, \ldots, T_g F_n)
\]
a.s. and therefore, by the boundedness of \( f \), we have convergence in \( L^2(\mathcal{F}^X) \). We conclude by

\[
T_g f(F_1, \ldots, F_n) = \lim_{l \to \infty} T_g f(F_{1,k_l}, \ldots, F_{n,k_l}) = \lim_{l \to \infty} f(T_g F_{1,k_l}, \ldots, T_g F_{n,k_l}) = f(T_g F_1, \ldots, T_g F_n),
\]

where the limits are taken in \( L^2(\mathcal{F}^X) \).

The Gaussian part of \( X \) is handled by the next lemma.

**Lemma 4.1.7.** Let \( g \in \mathbb{M}^{dyad} \) and let \( (\sigma B_t)_{t \in [0,1]} \) be the Brownian motion part of \( X \), where we assume that \( \sigma > 0 \) and that \( t \in (0,1] \) is dyadic. Then,

\[
T_g B_t = \int_{g((0,t])} dB_s \text{ a.s.}
\]

**Proof.** From the Lévy-Itô decomposition (Theorem 2.2.7) we know that there is a set \( \Omega_0 \) of measure one and a sequence \( (\alpha_N)_{N \in \mathbb{N}} \subseteq \mathbb{R} \), such that for all \( \omega \in \Omega_0 \), \( r \in [0,1] \), and \( E_N := (-N, -\frac{1}{N}) \cup (\frac{1}{N}, N) \), one has

\[
\sigma B_r(\omega) = X_r(\omega) - \lim_{N \to \infty} \left( \int_{\frac{1}{N} \times E_N} x \, dN(s,x) \right)(\omega) - \alpha_N r.
\]

Using the truncations \( \psi_L, L \in \mathbb{N} \), we get therefore

\[
\sigma B_t = \lim_{L \to \infty} \psi_L \left( X_t - \lim_{N \to \infty} \left( \int_{[0,t] \times E_N} x \, dN(s,x) \right) - \alpha_N t \right) \text{ a.s.,}
\]

\[
\sigma B_{g((0,t])} = \lim_{L \to \infty} \psi_L \left( X_{g((0,t])} - \lim_{N \to \infty} \left( \int_{g((0,t]) \times E_N} x \, dN(s,x) \right) - \alpha_N t \right) \text{ a.s.,}
\]

where we assume that \( g \) is represented by some fixed permutation of dyadic intervals and \( B_{g((0,t])} \) and \( X_{g((0,t])} \) are obtained by finite differences over these intervals in the canonical way. Moreover, the term \( \alpha_N t \) in the second equation appears due to the fact
that \( g \) is measure preserving. Therefore, it is sufficient to prove that
\[
T_g \psi_L \left( X_t - \lim_{N \to \infty} \int_{(0,t] \times E_N} x \, dN(s, x) - \alpha_N t \right)
= \psi_L \left( X_{g((0,t])} - \lim_{N \to \infty} \int_{g((0,t]) \times E_N} x \, dN(s, x) - \alpha_N t \right) \text{ a.s.}
\]
Because of the almost sure convergence in \( N \to \infty \) it is sufficient to verify that
\[
T_g \psi_L \left( X_t - \int_{(0,t] \times E_N} x \, dN(s, x) + \alpha_N t \right)
= \psi_L \left( X_{g((0,t])} - \int_{g((0,t]) \times E_N} x \, dN(s, x) + \alpha_N t \right) \text{ a.s.}
\]
for \( N \geq 2 \), or
\[
T_g \psi_L \left( \psi_K(X_t) - \int_{(0,t] \times E_N} x \, dN(s, x) + \alpha_N t \right)
= \psi_L \left( \psi_K(X_{g((0,t])}) - \int_{g((0,t]) \times E_N} x \, dN(s, x) + \alpha_N t \right) \text{ a.s.}
\]
for \( K, L \in \mathbb{N} \). As the integral terms belong to \( L^2(F^X) \), this follows from Lemmas 4.1.5 and 4.1.6.

### 4.1.2 Construction of \( S_g \)

For \( g \in M_{\text{dyad}} \) we define the operator
\[
S_g : \prod_{n=0}^{\infty} L^2_n \to \prod_{n=0}^{\infty} L^2_n \quad \text{by} \quad (f_n)_{n=0}^{\infty} \mapsto (S_{g,n}(f_n))_{n=0}^{\infty},
\]
where \( S_{g,n} : L^2_n \to L^2_n \) is given by
\[
f_n((-1,1), \ldots, (-1, x_n)) \mapsto f_n((g(t_1), x_1), \ldots, (g(t_n), x_n)).
\]
The distributions of \( f_n \) and \( S_{g,n} f_n \) coincide, so that the operators \( S_{g,n} \) and \( S_g \) are isometries. The next lemma shows that we can restrict ourselves to symmetric functionals in \( L^2_n \) when investigating \( S_g \).
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**Lemma 4.1.8** (Consistency with symmetrization).

1. For $f_n, h_n \in L^2_n$ with $I_n(f_n) = I_n(h_n)$ one has $I_n(S_{g,n} f_n) = I_n(S_{g,n} h_n)$.
2. For $f_n \in L^2_n$ one has $I_n(S_{g,n} f_n) = I_n(S_{g,n} \tilde{f}_n)$.

**Proof.** (2) follows from (1) by the property $I_n(f_n) = I_n(h_n)$ a.e. if and only if $\tilde{f}_n = \tilde{h}_n$ a.e. Hence it suffices to show that $\tilde{f}_n = \tilde{h}_n$ a.e. implies $\tilde{S}_{g,n} f_n = \tilde{S}_{g,n} h_n$ a.e. Using the transformation $r = g(s)$, this follows from

$$
\left( \tilde{S}_{g,n} f_n \right) ((s_1, x_1), \ldots, (s_n, x_n)) = \frac{1}{n!} \sum_{\varrho \in S_n} f_n ((g(s_\varrho(1)), x_\varrho(1)), \ldots, (g(s_\varrho(n)), x_\varrho(n)))
$$

for every $((r_1, x_1), \ldots, (r_n, x_n))$ for which $\tilde{f}_n$ and $\tilde{h}_n$ coincide. This concludes the proof.

**4.1.3 The commutative diagram**

**Theorem 4.1.9.** For $g \in \mathbb{M}^{dyad}$ the following diagram is commutative:

$$
\begin{array}{c}
L^2(\mathcal{F}^X) \xrightarrow{T_g} L^2(\mathcal{F}^X) \\
\jmath \uparrow \quad \uparrow \jmath \\
\bigoplus_{n=0}^{\infty} L^2_n \xrightarrow{S_{g^{-1}}} \bigoplus_{n=0}^{\infty} \tilde{L}^2_n \\
\end{array}
$$

(4.1.3)

**Proof.** As all linear combinations of

$$f_n((s_1, x_1), \ldots, (s_n, x_n)) = \mathbb{I}_{(a_i, b_i) \times E_i}(s_1, x_1) \cdot \cdots \cdot \mathbb{I}_{(a_n, b_n) \times E_n}(s_n, x_n),$$

where the $(a_1, b_1], \ldots, (a_n, b_n]$ are dyadic and pair-wise disjoint and the $E_i$ are of form $E_i = (c_i, d_i]$ with $c_i d_i > 0$ or $E_i = \{0\}$, are dense in $L^2_n$, and therefore the symmetrizations $f_n$ are dense in $\tilde{L}^2_n$, it suffices to show that $\jmath S_{g^{-1}}((0, \ldots, 0, \tilde{f}_n, 0, \ldots)) = T_g \jmath f_n$ for all $n \in \mathbb{N}$. For this it is sufficient to check that $I_n S_{g^{-1}, n} f_n = T_g I_n f_n$, which follows from Lemmas 4.1.5, 4.1.7, and 4.1.6, where we use that the sets $g((a_i, b_i])$ are pair-wise disjoint as well. 

$\square$

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4.2 Invariant sets

The invariance investigations in the following sections are closely related to classical ergodic theory. Therefore, these concepts shall be briefly presented and adapted to our setting. The main difference that we consider invariances with respect to a group of transformations whereas in the standard theory one considers mainly a single transformation operator and invariances with respect to the semigroup it generates, cf. [Shi96, Chapter V]).

We assume a measurable space \((S, \mathcal{S})\) and a group \(A\) of automorphisms of \(S\), i.e., a group of bijective bi-measurable functions \(T: S \to S\).

**Definition 4.2.1.** The invariant \(\sigma\)-algebra with respect to \(A\) is given by

\[ I(A) := \{ B \in \mathcal{S} : B = T^{-1}(B) \text{ for all } T \in A \} \]

It is easy to verify that \(I(A)\) is indeed a \(\sigma\)-algebra. Let \(A(B) := \bigcup_{T \in A} T(B)\). Another possibility for the definition of \(I(A)\) is the following.

**Lemma 4.2.2.** \(I(A) = \{ A(B) \in \mathcal{S} : B \in \mathcal{S} \}\).

**Proof.** We call the right-hand side \(I\). If \(B \in I\), we have that \(T^{-1}(B) = B\) for all \(T \in A\), therefore \(B \in I(A)\). On the other hand, if \(T^{-1}B = B\) for all \(T\) we have

\[ S \ni B = \bigcup_{T \in A} B = \bigcup_{T \in A} T^{-1}B = A(B). \]

**Lemma 4.2.3.** For a function \(\xi: S \to \mathbb{R}\) the following assertions are equivalent:

1. \(\xi\) is \(I(A)\)-measurable.
2. \(\xi\) is \(\mathcal{S}\)-measurable and constant on the orbits \(\{ Ts : T \in A \}, s \in S\).
3. \(\xi\) is \(\mathcal{S}\)-measurable and \(\xi \circ T = \xi\) for all \(T \in A\).

**Proof.** (2) \(\iff\) (3) is obvious. (1) \(\implies\) (3) Let \(\xi\) be a simple \(I(A)\)-measurable function

\[ \xi(s) := \sum_{k=1}^{N} \alpha_k \mathbb{1}_{B_k}(s) \text{ with } B_k \in I(A) \text{ and } \alpha_k \in \mathbb{R}. \]

In this case, the assertion follows from \(\mathbb{1}_{B_k}(T(s)) = \mathbb{1}_{T^{-1}(B_k)}(s) = \mathbb{1}_{B_k}(s)\). For a general \(I(A)\)-measurable \(\xi\) one finds a point-wise approximating sequence \(\xi_n\) of simple \(I(A)\)-measurable functions with \(\xi_n \circ T(s) = \xi_n(s)\) for all \(n \in \mathbb{N}, T \in A\) and \(s \in S\). This implies \(\xi \circ T(s) = \xi(s)\) by \(n \to \infty\).
4.2 Invariant sets

(2) \implies (1) We approximate \( \xi \) by

\[
\xi_n(s) := \sum_{k=-4^n}^{4^n-1} \frac{k}{2^n} \mathbb{1}_{B^*_k}(s) \quad \text{with} \quad B^*_k := \xi^{-1} \left( \left( \frac{k}{2^n}, \frac{k + 1}{2^n} \right) \right).
\]

As \( \xi \) is constant on the orbits, they are entirely contained in one of the sets \( B^*_k \), so that \( B^*_k \in \mathcal{I}(\mathbb{A}) \). Therefore \( \xi_n \) is \( \mathcal{I}(\mathbb{A}) \)-measurable and, by the point-wise convergence, \( \xi \) as well.

Let \( (S, \mathcal{S}, \gamma) \) be a \( \sigma \)-finite measure space with \( \gamma(S) > 0 \), \( \mathbb{A} \) be a group of automorphisms acting on \( S \), and

\[
\mathcal{I}(\mathbb{A}) := \mathcal{I}(\mathbb{A}) \lor \mathcal{N} \quad \text{where} \quad \mathcal{N} := \{ B \in \mathcal{S} : \gamma(B) = 0 \}.
\]

The equivalence class of \( \xi \) with respect to the \( \gamma \)-a.e.-equivalence is denoted by \([\xi]\).

**Definition 4.2.4.** The measure \( \gamma \) is called **quasi-invariant** with respect to \( \mathbb{A} \), if \( \gamma(T^{-1}B) = 0 \) for all \( B \in \mathcal{N} \) and \( T \in \mathbb{A} \).

**Lemma 4.2.5.** Let \( (S, \mathcal{S}, \gamma) \) be a \( \sigma \)-finite measure space with \( \gamma(S) > 0 \) and \( \mathbb{A} \) be a group of automorphisms acting on \( S \). Then one has the following assertions:

1. The operation \([\xi] \circ T := [\xi \circ T]\) is well-defined for all \( T \in \mathbb{A} \) and \( \mathcal{S} \)-measurable \( \xi : S \to \mathbb{R} \) if and only if \( \gamma \) is quasi-invariant with respect to \( \mathbb{A} \).
2. Let \( \gamma \) be quasi-invariant with respect to \( \mathbb{A} \) and \( \mathbb{A} \) be countable. Then \([\xi] \circ T = [\xi]\) for all \( T \in \mathbb{A} \) if and only if \( \xi \) is \( \mathcal{I}(\mathbb{A}) \)-measurable.

**Proof.** (1) Assume that \( \gamma \) is quasi-invariant and that \( \xi : S \to \mathbb{R} \) is \( \mathcal{S} \)-measurable. Then for \( \xi_1, \xi_2 \in [\xi] \) it holds that \( \gamma(\xi_1 \neq \xi_2) = 0 \) and the set

\[
\{ s : \xi_1(Ts) \neq \xi_2(Ts) \} = \{ T^{-1}t : \xi_1(t) \neq \xi_2(t) \}
\]

has measure zero as well so that the operator \([\xi] \mapsto [\xi] \circ T\) is well-defined. For the other implication let \( B \) be of measure zero and \( \xi := \mathbb{1}_B \) so that \([\xi] = 0\). By assumption, \([\xi \circ T] = 0\) and

\[
0 = \gamma(\{ s : \mathbb{1}_B(Ts) \neq 0 \}) = \gamma(T^{-1}(B)).
\]

(2) If there exists an \( \mathcal{I}(\mathbb{A}) \)-measurable \( \xi_0 \in [\xi] \) it is obvious that the equivalence class is invariant by (1) and Lemma 4.2.3. Conversely, let \([\xi \circ T] = [\xi]\) for all \( T \in \mathbb{A} \). Define

\[
S_0 := \{ s \in S : \xi \circ T(s) = \xi(s) \quad \text{for all} \quad T \in \mathbb{A} \} = \bigcap_{T \in \mathbb{A}} \{ s \in S : \xi \circ T(s) = \xi(s) \},
\]

which is a set of co-measure zero because \( \mathbb{A} \) is countable. Let us first prove that
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$S_0 \in \mathcal{I}(\mathcal{A})$. By definition, $S_0 \in \mathcal{S}$ and

$$T(S_0) = \{ t \in S : \xi \circ T'(s) = \xi(s) \text{ for all } T' \in \mathcal{A} \}$$

$$= \{ t \in S : \xi \circ T'(T^{-1}t) = \xi(T^{-1}t) \text{ for all } T' \in \mathcal{A} \}$$

$$= \{ t \in S : \xi \circ T'(T^{-1}t) = \xi(t) \text{ for all } T' \in \mathcal{A} \}$$

$$= \{ t \in S : \xi \circ T'(t) = \xi(t) \text{ for all } T' \in \mathcal{A} \}$$

$$= \{ t \in S : \xi \circ T'(t) = \xi(t) \text{ for all } T' \in \mathcal{A} \}$$

thus $S_0 \in \mathcal{I}(\mathcal{A})$. Setting $\xi_0(s) := \xi(s)\mathbb{1}_{S_0}(s)$, we obtain from Lemma 4.2.3 that $\xi_0$ is $\mathcal{I}(\mathcal{A})$-measurable and $\gamma$-a.e. equal to $\xi$.

**Definition 4.2.6.** Let $(S, \mathcal{I}, \gamma)$ be a $\sigma$-finite measure space with $\gamma(S) > 0$. A set $A \in \mathcal{I}$ with $\gamma(A) > 0$ is called **quasi-atom** provided that $B \subseteq A$ with $B \in \mathcal{I}$ implies that

$$\gamma(B) = 0 \quad \text{or} \quad \gamma(A \setminus B) = 0.$$

**Lemma 4.2.7.** Let $(S, \mathcal{I}, \gamma)$ be a $\sigma$-finite measure space with $\gamma(S) > 0$ and $A, A_1, A_2$ be quasi-atoms.

(1) If $B \in \mathcal{I}$ and $\gamma(A \Delta B) = 0$, then $B$ is a quasi-atom.

(2) If $A_1 \subseteq A_2$, then $\gamma(A_2 \setminus A_1) = 0$.

(3) Either $\gamma(A_1 \cap A_2) = 0$ or $\gamma(A_1 \Delta A_2) = 0$.

(4) There exist countably many pairwise disjoint quasi-atoms $(A_i)_{i \in I}$ such that $S \setminus (\bigcup_{i \in I} A_i)$ does not contain any quasi-atom. For any quasi-atom $A$ there is an $i \in I$ such that $\gamma(A \Delta A_i) = 0$.

**Proof.** The assumption $\sigma$-finite implies that all quasi-atoms have finite measure. Assertions (1), (2) and (3) are easy to prove and we skip the details.

(4a) First we assume that $\gamma$ is a finite measure. In this case we prove the statement in a constructive way. If there is no quasi-atom, then we are done. If there are quasi-atoms, then we take one with maximal measure (which exists) and call it $A_1$. Now we look for a quasi-atom $A_2 \subseteq A_1^c$. If there is no such atom, then the proof is again complete. If there is an atom, then we take one with maximal measure. We continue in this way. If the procedure stops, then we have a finite collection we are looking for. Assume now that we obtain an infinite sequence of disjoint quasi-atoms $(A_i)_{i=1}^{\infty}$ and let $A := \bigcup_{i=1}^{\infty} A_i$. We have to check that $A^c$ cannot contain any quasi-atom. Assume $B \subseteq A^c$ is a quasi-atom. Because $\lim_i \gamma(A_i) = 0$ there is some $i$ such that $\gamma(A_i) < \gamma(B)$. But this would be a contradiction to our construction.

(4b) In general, assume $S = \bigcup_{j \in J} S_j$ to be a disjoint union with $\gamma(S_j) \in (0, \infty)$.
4.3 Invariances for Lévy processes

We apply our construction to each $S_j$ and obtain a countable collection $(A_{i,j})_{i \in I, j \in J}$ of quasi-atoms where $I_j$ might be empty. Denote $R_j := S_j \setminus \bigcup_{i \in I_j} A_{i,j}$ and $R := \bigcup_{j \in J} R_j$. Assume that $A \subseteq R$ is a quasi-atom. Then there is exactly one index $j_A$ such that $\gamma(R_{j_A} \cap A) = \gamma(A)$. Letting $B := R_{j_A} \cap A$ gives that $\gamma(A \Delta B) = 0$ and that $B \subseteq R_{j_A}$ is a quasi-atom. But this is again a contradiction to our construction.

(4c) The remaining part of (4) is obvious.

**Lemma 4.2.8.** Let $(S, \mathcal{S}, \gamma)$ be a $\sigma$-finite measure space with $\gamma(S) > 0$ and $\mathcal{A}$ be a group of automorphisms of $S$ such that $(S, \mathcal{I}(A), \gamma)$ is $\sigma$-finite. Assume that $(A_i)_{i \in I}$ is a countable collection of quasi-atoms like in Lemma 4.2.7(4). Then for a function $\xi : S \to \mathbb{R}$ the following assertions are equivalent:

1. $\xi$ is $\mathcal{I}(A)$-measurable.
2. There exists a $\mathcal{S}$-measurable $\eta$ which is constant on the orbits and the quasi-atoms $(A_i)_{i \in I}$ and such that $\eta = \xi \gamma$-a.e.

**Proof.** (2) $\implies$ (1): Using Lemma 4.2.3 we get that $\eta$ is $\mathcal{I}(A)$-measurable, so that $\xi$ is $\mathcal{I}(A)$-measurable.

(1) $\implies$ (2): First we find an $\xi_0 \in [\xi]$ that is $\mathcal{I}(A)$-measurable. It can be easily seen that $\xi_0$ can be modified to an $\mathcal{I}(A)$-measurable random variable $\eta$ satisfying the claimed properties.

4.3 Invariances for Lévy processes

Throughout this section we let $G \subseteq \mathbb{M}^{\text{dyad}}$ be a subgroup of the group of dyadic measure preserving maps. For $n \in \mathbb{N}$ we derive the group $G[n]$ of the measure-preserving $((0,1] \times \mathbb{R})^n$-automorphisms

$$g[n] : ((t_1, x_1), \ldots, (t_n, x_n)) \mapsto (g(t_1), x_1), \ldots, (g(t_n), x_n))$$

with $g \in G$.

Now we introduce the main concepts of invariance we are interested in.

**Definition 4.3.1 (Notions of invariance).**

1. **$H_G$-invariance.** An $F \in L^2(\mathcal{F}^X)$ is $G$-invariant if $T_g F = F$ a.s. for all $g \in G$. The set of all $G$-invariant (equivalence classes of) random variables is denoted by $H_G$.

2. **$H_G$-measurability.** A symmetric chaos kernel $f_n : ((0,1] \times \mathbb{R})^n \to \mathbb{R}$ is $G[n]$-invariant if $f_n = f_n \circ g[n]$ a.e. for all $g \in G$. We let

$$\mathcal{H}_G := \sigma(\mathcal{I}_n(f_n) : f_n \text{ is } G[n]\text{-invariant}, \ n \in \mathbb{N}) \vee \mathcal{N}$$

with $\mathcal{N} := \{A \in \mathcal{F}^X : P(A) = 0\}$. 27
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(3) **$G$-invariant chaos expansion.** An $F \in L^2(F^X)$ has a $G$-invariant chaos expansion if all chaos kernels $f_n$ are symmetric and $G[n]$-invariant.

The definition of $H_G$ can understood in the way that we take particular representatives of $I_n(f_n)$ to define the $\sigma$-algebra. By adding the null-sets, all representatives become measurable with respect to $H_G$. The next theorem is the main result of this section:

**Theorem 4.3.2** (Invariance equivalences). For a group of dyadic measure preserving maps $G \subseteq M^\text{dyad}$ and $F \in L^2(F^X)$ the following assertions are equivalent:

1. $F \in H_G$.
2. $F$ is measurable with respect to $H_G$.
3. $F$ has a $G$-invariant chaos expansion.
4. $F$ has symmetric chaos kernels $f_n$, $n \in \mathbb{N}$, which are constant on the orbits of $G[n]$ on $((0,1] \times \mathbb{R})^n$.

**Definition 4.3.3.** If $F \in L^2(F^X)$ satisfies one of the equivalent conditions of Theorem 4.3.2, then we will say that $F$ is **$G$-invariant**.

In order to prove Theorem 4.3.2 we start with the following lemma.

**Lemma 4.3.4.** Let $F_1,\ldots,F_n \in H_G$ and $\varphi : \mathbb{R}^n \to \mathbb{R}$ be Borel measurable with $\varphi(F_1,\ldots,F_n) \in L^2(F^X)$. Then, $\varphi(F_1,\ldots,F_n) \in H_G$.

**Proof.** A Borel measurable function $\varphi$ can be approximated by truncation by bounded Borel measurable functions $\varphi_L := \psi_L(\varphi)$, $L \in \mathbb{N}$, and $\varphi_L(F_1,\ldots,F_n) \in H_G$ implies $\varphi(F_1,\ldots,F_n) \in H_G$ by monotone convergence and the completeness of $H_G$ (which is easy to check as the operators $T_g : L^2(F^X) \to L^2(F^X)$ are isometries).

Assuming that $\varphi$ is bounded, we approximate $\varphi$ point-wise by simple functions $\varphi_k$ with $\|\varphi_k\|_\infty \leq \|\varphi\|_\infty$. It follows that $\varphi_k(F_1,\ldots,F_n) \to \varphi(F_1,\ldots,F_n)$ in $L^2(F^X)$ by dominated convergence. Therefore, it is sufficient to check the statement for $\varphi = \mathbbm{1}_B$ where $B$ is a Borel set from $\mathbb{R}^n$. Using the outer regularity of the law of $(F_1,\ldots,F_n)$ we can verify this using $\varphi \in C^b(\mathbb{R}^n)$. But this case follows from Lemma 4.1.6. \qed

**Proof of Theorem 4.3.2.** (1) $\iff$ (3) follows from Theorem 4.1.9 and the uniqueness of symmetric kernels in the chaos expansion.

(3) $\implies$ (2) follows by definition and the completeness of $(\Omega, H_G, P)$.

(4) $\implies$ (3) is a consequence of Lemma 4.2.3.

(3) $\implies$ (4) First we use Lemma 4.2.5 to obtain a chaos kernel that is constant on the orbits. This new kernel will be symmetrized which keeps the property that the kernel is constant on the orbits.
4.4 Diagonal groups and locally ergodic sets

(2) \implies (1) As $H_G$ is a closed subspace of $L^2(F^X)$, it is sufficient to check that $1_A \in H_G$ for all $A \in H_G$. Here it is sufficient to take $A$ such that there exists a sequence $(I_{i_k}(f_{i_k}))_{k \in \mathbb{N}}$ with $(i_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$ and $G[i_k]$-invariant kernels $f_{i_k}$ such that $A \in G := \sigma(I_{i_k}(f_{i_k}) : k \in \mathbb{N})$.

By martingale convergence, $1_A$ can be approximated in $L^2$ by $G_n$-measurable functions, where $G_n := \sigma(I_{i_k}(f_{i_k}) : k \in \{1, \ldots, n\})$.

By Doob’s factorization lemma (cf. [Bau01, Lemma II.11.7]), there are Borel functions $\varphi_n : \mathbb{R}^n \to \mathbb{R}$ such that $E[1_A | G_n] = \varphi_n(I_{i_1}(f_{i_1}), \ldots, I_{i_n}(f_{i_n}))$ a.s., so that $\lim_n E[|1_A - \varphi_n(I_{i_1}(f_{i_1}), \ldots, I_{i_n}(f_{i_n}))|^2] = 0$.

Because of the equivalence $(1) \iff (3)$ we have that $I_{i_k}(f_{i_k}) \in H_G$ for all $k \in \mathbb{N}$ and Lemma 4.3.4 implies that $\varphi_n(I_{i_1}(f_{i_1}), \ldots, I_{i_n}(f_{i_n})) \in H_G$.

Because $H_G$ is closed in $L^2(F^X)$, we derive that $1_A \in H_G$. \hfill \Box

## 4.4 Diagonal groups and locally ergodic sets

Let $(T, \mathcal{T}, \tau, (T_N)_{N=0}^\infty)$ be a filtered probability space such that there are refining partitions
\[ T = T_{N,1} \cup \cdots \cup T_{N,L_N}, \quad N = 0, 1, 2, \ldots, \]
satisfying the following assumptions:

1. $\mathcal{T}_N = \sigma(T_{N,1}, \ldots, T_{N,L_N})$,
2. $\tau(T_{N,I}) > 0$ for all $(N,I)$,
3. $\lim_{N \to \infty} \sup_{l=1, \ldots, L_N} \tau(T_{N,l}) = 0$ and
4. $\mathcal{T} = \bigvee_{N=0}^\infty \mathcal{T}_N$.

We let $\mathcal{O}(T)$ be the system of countable unions of elements from $\bigcup_{N=0}^\infty \mathcal{T}_N$ (including the empty set). The system forms a topology. In particular, a set $G \subseteq T$ is open provided that it is empty or for each $x \in G$ there is a $T_{N,I}$ with $x \in T_{N,I} \subseteq G$.\hfill 29
Finally, we suppose that there is a countable group $G$ of bijective bi-measurable automorphisms $g: T \to T$. If $G \subseteq \mathcal{M}^{\text{dyad}}$ and $A \subseteq T$, the restriction $G|_A$ consists of all restrictions of $g \in G$ to $A$ such that $g|_A: A \to T$ is an automorphism of $A$ (in particular, it has range in $A$).

**Definition 4.4.1.**

(1) A set $E \subseteq T$ of positive measure is called **finitely locally ergodic** with respect to $G$ provided that there is an $N_E \geq 0$ such that $E \in \mathcal{T}_{N_E}$ and for all $A := T_{N,l} \cup T_{N,m} \subseteq E$ with $l \neq m$ and $N \geq N_E$ there is a subgroup $H \subseteq G$ such that

a) $g|_A = \text{id}_A$ for all $g \in H$,

b) the probability space $(A, \mathcal{I}(H|_A), \tau_A)$ is trivial, i.e. contains only sets of measure one or zero, where $\tau_A$ the normalized restriction of $\tau$ to $A$.

(2) A set $E \subseteq T$ is called **locally ergodic** with respect to $G$ provided that there is a sequence $E^j$ of finitely locally ergodic sets with respect to $G$ such that

$$E^1 \subseteq E^2 \subseteq \cdots \subseteq E \quad \text{and} \quad E = \bigcup_{j=1}^{\infty} E^j.$$

**Remark 4.4.2.**

(1) By definition, locally ergodic sets belong to $\mathcal{O}(T)$.

(2) Local ergodicity is stable with respect to passing to open subsets: If $\emptyset \neq F \subseteq E$, where $F \in \mathcal{O}(T)$ and where $E$ is locally ergodic, then $F$ is locally ergodic.

**Proof.** Let us check (2). By definition, we find finitely locally ergodic sets such that

$$E^1 \subseteq E^2 \subseteq \cdots \subseteq E \quad \text{and} \quad E = \bigcup_{j=1}^{\infty} E^j.$$

At the same time we find an increasing sequence $F^j \in \mathcal{T}_{N,j}$, $j \in \mathbb{N}$, such that $F = \bigcup_{j=1}^{\infty} F^j$. One obtains

$$F = F \cap E = \bigcup_{j=1}^{\infty} (F^j \cap E^j)$$

and that $F^j \cap E^j$ is finitely locally ergodic because $E^j$ is of this type and $F^j \cap E^j \subseteq E^j$.

Now we define the **diagonal group** which fits the context of chaos expansions: We fix $n \in \mathbb{N}$ and consider an auxiliary $\sigma$-finite measure space $(R, \mathcal{R}, \rho)$ with $\rho(R) > 0$ and the group $G[n]$ that consists of all maps $g[n]: (T \times R)^n \to (T \times R)^n$ given by

$$((t_1, x_1), \ldots, (t_n, x_n)) \to ((g_1(t_1), x_1), \ldots, (g(t_n), x_n)) \quad \text{with} \quad g \in G.$$
This definition is consistent with the operator $S_{g,n}$ defined above: $S_{g,n} f = f_n \circ g[n]$. To formulate the main result for locally ergodic sets, we recall that $\mathcal{I}(\mathbb{G}[n])$ denotes the invariant $\sigma$-algebra with respect to the group $\mathbb{G}[n]$, see Definition 4.2.1. For $A \in \mathcal{T}$ the trace $\sigma$-algebra on $A$ is denoted by $\mathcal{T}|_A$.

**Theorem 4.4.3.** Let $n \in \mathbb{N}$, $E_1, \ldots, E_L \in \mathcal{T}$ be pairwise disjoint and locally ergodic with respect to $\mathbb{G}$, let

$$\mathcal{T}_E := \mathcal{T}|_{\mathcal{T} \setminus \bigcup_{l=1}^L E_l} \lor \sigma(E_1, \ldots, E_L),$$

and

$$\mathcal{N}_n := \{ A \in (\mathcal{T} \otimes \mathcal{R})^\otimes n : (\tau \otimes \rho)^\otimes n(A) = 0 \}.$$ 

Then $\mathcal{I}(\mathbb{G}[n]) \subseteq (\mathcal{T}_E \otimes \mathcal{R})^\otimes n \lor \mathcal{N}_n$.

**Lemma 4.4.4.** Assume a probability space $(\mathcal{M}, \mathcal{M}, m)$, a decreasing sequence of measurable sets $D_0 \supseteq D_1 \supseteq \ldots$, a sub-$\sigma$-algebra $I \subseteq \mathcal{M}$ and

$$\mathcal{G}_N := \mathcal{I} \lor \sigma(A_N \in \mathcal{M} \text{ with } A_N \subseteq D_N).$$

Assume that $m(D_N) \to 0$ as $N \to \infty$. Then

$$\bigcap_{N=0}^\infty \mathcal{G}_N \lor \mathcal{N} \subseteq \mathcal{T} \lor \mathcal{N} \text{ with } \mathcal{N} := \{ A \in \mathcal{M} : m(A) = 0 \}.$$ 

**Proof.** The $\sigma$-algebra $\mathcal{G}_N$ consists of all

$$B_N = (I_N \cap D_N^c) \cup A_N$$

with $A_N \in \mathcal{M}$, $A_N \subseteq D_N$ and $I_N \in \mathcal{I}$. Therefore $B \in \bigcap_{N=0}^\infty (\mathcal{G}_N \lor \mathcal{N})$ gives $I_N \in \mathcal{I}$ and $A_N \in \mathcal{M}$ with $A_N \subseteq D_N$ such that

$$B_N := (I_N \cap D_N^c) \cup A_N \text{ satisfies } B_N \Delta B \in \mathcal{N} \text{ for all } N \geq 0.$$ 

Defining $C := \bigcup_{N=0}^\infty (B_N \Delta B) \in \mathcal{N}$, this implies on $\mathcal{C}$ that

$$B = B_N = (I_N \cap D_N^c) \cup A_N.$$

Let

$$I := \bigcup_{N=0}^\infty \bigcap_{k=N}^\infty I_k \in \mathcal{I}.$$ 

By construction, $I_N = B_N$ on $D_N^c$ and $D_0 \subseteq D_1^c \subseteq \cdots$. Therefore, $I \Delta B \subseteq D_N \cup C$ which implies $\mathbb{P}(I \Delta B) \leq \lim_N \mathbb{P}(D_N) = 0$ and proves the lemma. \qed
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Proof of Theorem 4.4.3. We assume a partition \( R = \bigcup_{j \in J} R_j \) with \( \rho(R_j) \in (0, \infty) \). Choosing \( \lambda_j \in (0, \infty) \) we can arrange that \( \rho^0(A) := \sum_{j \in J} \lambda_j \rho(A \cap R_j) \) becomes a probability measure which has a strictly positive density with respect to \( \rho \). As our statement only concerns null sets we can replace \( \rho \) by \( \rho^0 \), or we can assume without loss of generality that \( \rho \) itself is a probability measure.

1. First we assume that \( E_1, \ldots, E_L \) are finitely locally ergodic. Let us fix a set \( B \in \mathcal{I}(\mathcal{G}[n]) \) of positive measure.

(a) We observe that \( \bigvee_{n \geq 0} (T_N \otimes \mathcal{R})^{\otimes n} = (\mathcal{T} \otimes \mathcal{R})^{\otimes n} \), so that martingale convergence yields

\[
\lim_{N \to \infty} f_N = \mathbb{1}_B \ (\tau \otimes \rho)^{\otimes n} \text{-a.s.,}
\]

where, for \( (t_1, \ldots, t_n) \in Q^N_{i_1, \ldots, i_n} := T_{N,i_1} \times \cdots \times T_{N,i_n} \), we define

\[
f_N((t_1, x_1), \ldots, (t_n, x_n)) := \int_{Q^N_{i_1, \ldots, i_n}} \mathbb{1}_B((s_1, x_1), \ldots, (s_n, x_n)) \, d\tau(s_1) \cdots d\tau(s_n) / \tau^{\otimes n}(Q^N_{i_1, \ldots, i_n}).
\]

(b) For \( N \geq 0 \) we let

\[
\Delta_N := \bigcup_{i_1, \ldots, i_n \in \{1, \ldots, L_N\}} Q^N_{i_1, \ldots, i_n}
\]

which is empty for \( n = 1 \). For \( n \geq 2 \) the size of \( \Delta_N \) can be upper bounded by

\[
\tau^{\otimes n}(\Delta_N) \leq \binom{N}{2} \max_{i=1, \ldots, N} \tau(T_{N,i}) \quad \text{so that} \quad \lim_N \tau^{\otimes n}(\Delta_N) = 0.
\]

Define

\[
\mathcal{G}_N := (\mathcal{T}_E \otimes \mathcal{R})^{\otimes n} \vee \sigma \left( D \times G : D \in \mathcal{T}^{\otimes n}, D \subseteq \Delta_N, G \in \mathcal{R}^{\otimes n} \right)
\]

with a slight abuse of notation concerning the order of components, which gives the \( \sigma \)-algebra \( \mathcal{T}_E \otimes \mathcal{R} \) in the case \( n = 1 \). As \( \Delta_0 \supseteq \Delta_1 \supseteq \cdots \) we have \( \mathcal{G}_0 \supseteq \mathcal{G}_1 \supseteq \cdots \).

(c) Let \( N_0 := \max\{N_{E_1}, \ldots, N_{E_L}\} \geq 0 \), where the indices \( N_{E_l} \) are taken from Definition 4.4.1 (1). The main observation of the proof is that \( f_N \in \mathcal{G}_N \)-measurable for \( N \geq N_0 \). By definition, \( f_N \) is constant on all cuboids \( Q^N_{i_1, \ldots, i_n} \). Assume two cuboids

\[
Q^N_{i_1, \ldots, i_n} \quad \text{and} \quad Q^N_{m_1, l_2, \ldots, l_n}
\]

such that \( (l_1, \ldots, l_n) \) are distinct, \( (m_1, l_2, \ldots, l_n) \) are distinct, \( l_1 \neq m_1 \), and that \( T_{N,i_1}, T_{N,m_1} \subseteq E_l \), where \( l \in \{1, \ldots, L\} \) is now fixed. By assumption, there is a subgroup \( \mathcal{H} \) of \( G \) such that for \( A_l := T_{N,i_1} \cup T_{N,m_1} \), the probability space \( (A_l, \mathcal{I}(\mathcal{H} | A_l), \tau_{A_l}) \)
4.4 Diagonal groups and locally ergodic sets

is trivial and $\mathbb{I}$ acts as an identity outside $A_1$. Because $B \in \mathcal{I}(\mathbb{G}[n])$ we have that

$$\mathbb{I}_{B}g[n] = \mathbb{I}_{B} \quad \text{for all } g \in \mathbb{G},$$

so that, for all $g \in \mathbb{I}$,

$$\mathbb{I}_{B}((gt_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)) = \mathbb{I}_{B}((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n))$$
on $(A_1 \times R) \times (T_{N,t_2} \times R) \times \cdots \times (T_{N,t_n} \times R)$. This implies that the subset $A_1$ of the section of $B$, taken at

$$(x_1, (t_2, x_2), \ldots, (t_n, x_n)) \in R \times (T_{N,t_2} \times R) \times \cdots \times (T_{N,t_n} \times R), \quad (4.4.1)$$
is invariant with respect to $\mathbb{I}_{A_1}$ and therefore the function

$$t_1 \to \mathbb{I}_{B}((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n))$$
is almost surely constant on $A_1$ under the condition (4.4.1). Consequently,

$$\int_{Q^n_{m_1,\ldots,m_n}} \mathbb{I}_{A}((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)) \frac{d\tau(t_1) \cdots d\tau(t_n)}{\tau \otimes^n (Q^n_{m_1,\ldots,m_n})} = \int_{Q^n_{m_1,\ldots,m_n}} \mathbb{I}_{A}((t_1, x_1), (t_2, x_2), \ldots, (t_n, x_n)) \frac{d\tau(t_1) \cdots d\tau(t_n)}{\tau \otimes^n (Q^n_{m_1,\ldots,m_n})}$$

for $(x_1, \ldots, x_n) \in R^n$. We can repeat the argument, where we replace the exchange of the first component of the cuboid by any other component. This implies that $f_N$ is $\mathcal{G}_N$-measurable.

(d) From (c) we immediately get that $f_M$ is $\mathcal{G}_N$-measurable for $M \geq N \geq N_0$. Therefore $\mathbb{I}_{B}$ is $\mathcal{G}_N \vee \mathcal{N}_n$-measurable for all $N \geq N_0$. Applying Lemma 4.4.4 we get that $\mathbb{I}_{B}$ is $(\mathcal{T}_E \otimes \mathcal{R}) \otimes^n \mathcal{N}_n$-measurable.

II. Now we assume general locally ergodic sets $E_1, \ldots, E_L$. By definition, we find monotone sequences of finitely locally ergodic sets $(E^j_i)_{j=1}^\infty$ with

$$\bigcup_{j=1}^\infty E^j_i = E_i.$$

We proved in step I that $\mathcal{I}(\mathbb{G}[n]) \subseteq (\mathcal{T}_{E_i} \otimes \mathcal{R}) \otimes^n \mathcal{N}_n$ with

$$\mathcal{T}_{E_i} := \mathcal{T}_{\mathcal{T}(\bigcup_{j=1}^{\infty} E^j_i) \setminus \sigma(E^1_i, \ldots, E^L_i)},$$
so that
\[ I(G[n]) \subseteq \bigcap_{j=1}^{\infty} \left( (T_{E_j} \otimes \mathcal{R})^\otimes n \vee \mathcal{N}_n \right). \]

Observing
\[ (T_{E_j} \otimes \mathcal{R})^\otimes n \subseteq (T_{E} \otimes \mathcal{R})^\otimes n \vee \sigma(A^j \in (T \otimes \mathcal{R})^\otimes n : A^j \subseteq D^j) \]
with
\[ D^j := \left\{ (t_1, x_1), \ldots, (t_n, x_n) \in (T \times R)^n : t_k \in \bigcup_{l=1}^{L}(E_l \setminus E_l^j) \text{ for some } k \in \{1, \ldots, n\} \right\} , \]
gives that
\[ I(G[n]) \subseteq \bigcap_{j=1}^{\infty} \left( (T_{E} \otimes \mathcal{R})^\otimes n \vee \sigma(A^j \in (T \otimes \mathcal{R})^\otimes n : A^j \subseteq D^j) \vee \mathcal{N}_n \right). \]

Finally, because of \( D^1 \supseteq D^2 \supseteq \cdots \) and
\[ (\tau \otimes \rho)^\otimes n (D^j) \leq n \left[ \sum_{l=1}^{L} \tau(E_l \setminus E_l^j) \right] \rightarrow 0 \text{ as } j \rightarrow \infty, \]
we can again apply Lemma 4.4.4 which concludes the proof. \( \square \)

### 4.5 Reduced chaos expansions

In this section we apply the results from Section 4.4 to Lévy processes. For this purpose we let

1. \((T, T, \tau) := ((0, 1], \mathcal{B}((0, 1]], \lambda) \) with
\[ T_N = \mathcal{F}_N^\text{dyad} := \sigma \left( \left( \left( \frac{l-1}{2^N}, \frac{l}{2^N} \right) : l = 1, \ldots, 2^N \right) \right), \]
2. \([E]_\text{dyad} := \{ g \in [E]^\text{dyad} : g|_{E^c} = \text{id}_{E^c} \} \) for \( E \subseteq (0, 1], \)
3. \((R, \mathcal{R}, \rho) := (\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu), \)
4. and \( \mathcal{N}_n \) be the null-sets in \(((0, 1] \times \mathbb{R})^n \) with respect to \( \mathfrak{m}_n^\otimes n = (\lambda \otimes \mu)^\otimes n. \)

Let us begin with a prototype of a locally ergodic set.
4.5 Reduced chaos expansions

**Lemma 4.5.1.** Let \( E \in \mathcal{O}((0,1]) \) be non-empty. Then \( E \) is locally ergodic with respect to \( \mathcal{M}_d^{\text{dyad}} \).

**Proof.** It is enough to show the following: If \( A \in \mathcal{F}_d^{\text{dyad}} \) is a non-empty subset of \( E \), then \((A, \mathcal{I}(\mathcal{M}_d^{\text{dyad}}|_A), \lambda_A)\) is trivial. Take any \( B \in \mathcal{I}(\mathcal{M}_d^{\text{dyad}}|_A) \). Using the dyadic filtration restricted to \( A \), where we start with the level \( N_0 \), we interpret \( \mathbb{1}_B \) as closure of a martingale in \((A, \mathcal{B}((0,1])|_A, \lambda_A)\) along this filtration. By the invariance of \( B \), the random variables, that form this martingale, are individually constant. Therefore we get a sequence of constants that converge to \( \mathbb{1}_B \) in \( L^2(A, \lambda_A) \) and \( \lambda_A \)-a.s. Hence \( \mathbb{1}_B \) is a constant almost surely which implies the statement. \( \square \)

**Remark 4.5.2.** One can find groups \( G \) such that for example \( E = (0,1] \) is locally ergodic but \( G \subseteq \mathcal{M}_d^{\text{dyad}} \). Take for example all permutations that leave the first interval \((0, 2^{-N}] \) invariant on each dyadic level \( N \). It would be of interest to characterize those sub-groups \( G \subseteq \mathcal{M}_d^{\text{dyad}} \) such that a given \( E \in \mathcal{O}((0,1]) \) gets locally ergodic.

Now we let \( G \) be a group like in Section 4.3. The main result is the following simplification of the chaos decomposition:

**Theorem 4.5.3 (Locally ergodic expansion).** For pair-wise disjoint \( E_1, \ldots, E_L \in \mathcal{O}((0,1]) \), that are locally ergodic with respect to \( G \), and \( F \in L^2(\mathcal{F}^X) \) consider the following conditions:

1. \( F \) is \( G \)-invariant.
2. One has \( F = \sum_{n=0}^{\infty} I_n(f_n) \) with symmetric \( (\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n \) \( \mathcal{N}_n \)-measurable and \( G[n] \)-invariant chaos kernels \( f_n \).
3. One has \( F = \sum_{n=0}^{\infty} I_n(f_n) \) with symmetric \( (\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n \)-measurable chaos kernels \( f_n \).
4. \( F \) is invariant with respect to the group \( H \) generated by \( \mathcal{M}^{\text{dyad}}_{E_1}, \ldots, \mathcal{M}^{\text{dyad}}_{E_L} \).
5. \( F \) is measurable with respect to \( \mathcal{A}_E : = \sigma(I_1(f_1) : f_1 \in L^2_1) \) is \( \mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}) \)-measurable \( \mathcal{N} \).

Then it holds that \( (1) \iff (2) \implies (3) \iff (4) \iff (5) \). Moreover, for \( G = H \) all assertions are equivalent.

For the proof we need the following observation which is inspired by the product formula of Lee and Shih, cf. [LS04].

**Lemma 4.5.4.** Let \( H \) be a group of dyadic permutations of \((0,1]\). For \( n, m \geq 1, 0 \leq k \leq n \wedge m, 0 \leq r \leq (n \wedge m) - k \), \( f \in L^2_n \), and \( f' \in L^2_m \) we define \( f \otimes_k f' : (\mathbb{R}^{n-k-r} \times (0,1]) \otimes (\mathbb{R}^{m-k-r} \times (0,1]) \rightarrow \mathbb{R} \)
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by

\[ (f \otimes_k f')(\alpha, \beta, \gamma) = \Pi_x(\gamma) \int_{((0,1] \times \mathbb{R})^k} f(\alpha, \gamma, \rho) f'(\rho, \gamma, \beta) \, \text{d}\mathcal{m} \otimes^k \rho \]

where \( \Pi_x(\gamma) \) is the product of the x-coordinates of the vector \( \gamma \) and where we assume that

\[ \int_{((0,1] \times \mathbb{R})^k} |f(\alpha, \gamma, \rho) f'(\rho, \gamma, \beta)| \, \text{d}\mathcal{m} \otimes^k \rho < \infty \]

for all \((\alpha, \beta, \gamma) \in ((0,1] \times \mathbb{R})^{n+m-2k-r}\). If \( f \) is constant on the orbits of \( \mathbb{H}[n] \) and \( g \) is constant on the orbits of \( \mathbb{H}[m] \), then \( f \otimes_k g \) is constant on the orbits of \( \mathbb{H}[n+m-2k-r] \).

**Proof.** For \( g \in \mathbb{H} \) we simply obtain that

\[
(f \otimes_k f')(g[n - k - r]\alpha, g[m - k - r]\beta, g[r]\gamma) = \int_{((0,1] \times \mathbb{R})^k} f(g[n - k - r]\alpha, g[r]\gamma, \rho) f'(\rho, g[r]\gamma, g[m - k - r]\beta) \, \text{d}\mathcal{m} \otimes^k \rho = \int_{((0,1] \times \mathbb{R})^k} f(\alpha, \gamma, \rho) f'(\rho, \gamma, \beta) \, \text{d}\mathcal{m} \otimes^k \rho = (f \otimes_k f')(\alpha, \beta, \gamma). \]

For the proof of Theorem 4.5.3 we denote by \((f \otimes_k^g g)\) the symmetrization of \((f \otimes_k g)\), and by \(f_1 \otimes \cdots \otimes f_n\) the symmetrization of \(f_1 \otimes \cdots \otimes f_n\).

**Proof of Theorem 4.5.3.** \( (2) \implies (1) \) follows from Theorem 4.3.2 and \( (1) \implies (2) \) from Theorems 4.3.2 and 4.4.3.

\( (2) \implies (3) \) We find a \((\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n\)-measurable \( f_n' \), with \( f_n' = f_n \), a.e. By symmetrizing this \( f_n' \), we get a symmetric kernel that is \((\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n\)-measurable.

\( (3) \implies (4) \) follows again from Theorem 4.3.2.

\( (4) \implies (3) \) By Theorem 4.3.2, we get symmetric and \( \mathbb{H}[n] \)-invariant kernels. On the other side, Lemma 4.5.1 yields that \( E_1, \ldots, E_L \) are locally ergodic with respect to \( \mathbb{H} \) so that \( \mathcal{I}[\mathbb{H}[n]] \subseteq (\mathcal{B}((0,1])_E \otimes \mathcal{B}(\mathbb{R}))^\otimes n \lor \mathcal{N}_n \) by Theorem 4.4.3. One can finish as in \( (2) \implies (3) \).

\( (5) \implies (4) \) From Lemma 4.5.1 we know that \( E_1, \ldots, E_L \) are locally ergodic with respect to \( \mathbb{H} \). Next we observe that \( I_1(f_1) \) is \( \mathbb{H} \)-invariant so that (using the arguments from (1) \( \iff (2) \) and the a.e. uniqueness of \( f_1 \) one can replace \( f_1 \) by an \( \mathbb{H}[1] \)-invariant kernel \( f_1' \). Therefore, \( F \) is \( \mathcal{H}_{\mathbb{H}} \)-measurable.

\( (3) \implies (5) \) Let \( m \geq 1 \) and \( f_0, \ldots, f_m \in L^2_{\mathbb{H}} \) be step-functions based on sets of type \( A \times J \) with \( A \in \{ E_1, \ldots, E_L \} \) or \( A \subseteq (E_1 \cup \cdots \cup E_L)^c \) is a Borel set and \( J = (a, b] \) or \( J = \{-b, -a\} \) with \( 0 < a < b < \infty \), or \( J = \{0\} \). Then the \( f_i \) are constant on the orbits.
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of $\mathbb{H}[1]$ and their integrability assures that we can we apply [LS04, Theorem 3.5] to get that

$$I_{m+1}(f_0 \otimes f_1 \otimes \cdots \otimes f_m) =$$

$$I_1(f_0)I_m(f_1 \otimes \cdots \otimes f_m) - m [I_m(f_0 \otimes f_1 \otimes \cdots \otimes f_m) + I_{m-1}(f_0 \otimes f_1 \otimes \cdots \otimes f_m)].$$

Because of Lemma 4.5.4, all integrands occurring on the right-hand side are constant on the orbits of $\mathbb{H}[1], \mathbb{H}[m], \mathbb{H}[m], \mathbb{H}[m-1]$, respectively. This implies that $I_{m+1}(f_0 \otimes f_1 \otimes \cdots \otimes f_m)$ is measurable with respect to

$$\mathcal{H}^{(m)} := \sigma(I_n(h_n) : h_n \text{ is } \mathbb{H}[n] - \text{invariant, } n \in \{1, \ldots, m\}) \setminus \mathcal{N}.$$  

Let $h_{m+1}$ be symmetric and $(B((0, 1])_E \otimes B(\mathbb{R}))^{\otimes (m+1)}$-measurable. It is standard that finite linear combinations of tensor products $f_0 \otimes f_1 \otimes \cdots \otimes f_m$ of the above form can be used to approximate $h_{m+1}$ in $L^2_{m+1}$. Using that any $L^2$-convergent sequence contains a sequence that converges almost surely, we get that $I_{m+1}(h_{m+1})$ is already $\mathcal{H}^{(m)}$-measurable. Induction over $m$ and the identity $\mathcal{H}^{(1)} = \mathcal{A}_E$ yield the implication (3) $\implies$ (5).

Finally, the equivalence of the assertions in case $\mathbb{H} = \mathbb{G}$ is obvious as $E_1, \ldots, E_L$ are locally ergodic with respect to $\mathbb{H}$ as already used above.

**Remark 4.5.5.**

(1) If $E_1, \ldots, E_L$ from Theorem 4.5.3 form a partition of $(0, 1]$, then the symmetric kernels $f_n$ in Theorem 4.5.3 (3) are constant in the time variables on all cuboids $E_{l_1} \times \cdots \times E_{l_n}$ with $l_1, \ldots, l_n \in \{1, \ldots, L\}$.

(2) Given a system of $B((0, 1])_E \otimes B(\mathbb{R})$-measurable $f_1', f_2', \ldots$ such that

$$\mathcal{A}_E = \sigma(I_l(f_l') : l = 1, 2, \ldots) \setminus \mathcal{N},$$

Theorem 4.5.3 (5) is equivalent to the fact that we find a functional $\Phi : \mathbb{R}^N \to \mathbb{R}$, measurable with respect to the Borel $\sigma$-algebra on $\mathbb{R}^N$ generated by the cylinder sets, such that

$$F = \Phi(I_1(f_1'), I_1(f_2'), \ldots) \text{ a.s.}$$

This follows from a factorization due to Doob, cf. [Bau01, Lemma II.11.7]. For example, for $E = (0, 1]$, this leads to representations of $F$ in terms of $B_1$ (the normalized Brownian part if present) and $N((0, 1], (a, b))$ with $ab > 0$.

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4.6 Examples

4.6.1 A negative example: Shift operators

The following example indicates that in spite of applicability of Theorem 4.3.2, a componentwise reduction as in Theorem 4.5.3 is not always possible and motivate thus the need of locally ergodic sets. We do this by considering the group generated by shifts, which is inspired by the work of Itô [Itô56]. Let us start with a positive remark.

Example 4.6.1. A functional \( F \in L^2(F^X) \) is invariant with respect to all dyadic periodic shifts if and only if \( F \) is measurable with respect to

\[
\mathcal{H}_{\text{shift}} := \sigma \left( I_n(f_n) : f_n \text{ symm. and } f_n = f_n \circ s_l[n], l, n \in \mathbb{N} \right) \vee \{ A \in F^X : \mathbb{P}(A) = 0 \},
\]

where \( s_l : (0, 1] \to (0, 1] \) is the periodic shift to the right of length \( 2^{-l} \) of the interval \( (0, 1] \).

Now we show that a componentwise reduction is not possible. Assume that

\[
F = I_2(f_2) \quad \text{where} \quad f_2((s,x),(t,y)) := g_2(|s-t|)h_2(x,y)
\]

with a measurable function \( g_2 : [0,1] \to \mathbb{R} \) such that \( g_2(1/2 - s) = g_2(1/2 + s) \) for \( s \in [0, 1/2] \) and a symmetric Borel function \( h_2 : \mathbb{R}^2 \to \mathbb{R} \) such that \( f_2 \in L_2^2 \). It is straightforward to check that \( F \) is invariant with respect to all shifts \( s_h : (0, 1] \to (0, 1] \), \( 0 < h < 1 \), defined by \( s_h(t) := t + h \) if \( t + h \leq 1 \) and \( s_h(t) := t + h - 1 \) if \( t + h > 1 \). Obviously, the measure \( \mu \) and the functions \( g_2 \) and \( h_2 \) can be chosen such that there is no symmetric \( f_2((s,x),(t,y)) \) not depending on \( (s,t) \), but with \( f_2 = \bar{f}_2 \) a.e. (take for example \( \mu \) as the Dirac measure in 1).

4.6.2 Positive examples

The positive examples are based on Proposition 4.6.3 below for which we need the notion of weak \( G \)-invariance:

Definition 4.6.2. Given a subgroup \( G \subseteq M^{dyad} \), we say that an \( F^X \)-measurable random variable \( Z : \Omega \to \mathbb{R} \) is weakly \( G \)-invariant provided that \( f(Z) \) is \( G \)-invariant for all \( f \in C_b(\mathbb{R}) \).

\( G \)-invariance implies weak \( G \)-invariance by Lemma 4.1.6, but the converse does not need to be true because of a possibly missing integrability. To consider our examples,
let us fix a sequence of time-points

\[ 0 \leq s_1 < t_1 \leq \ldots \leq s_L < t_L \leq 1 \]

together with the corresponding intervals \( E_l := (s_l, t_l] \) for the rest of this section. Similarly as before, we let

\[ \mathcal{B}((0, 1])_E := \mathcal{B} \left( (0, 1] \setminus \bigcup_{l=1}^L (s_l, t_l] \right) \lor \sigma((s_1, t_1], \ldots, (s_L, t_L)). \]

**Proposition 4.6.3.** Assume \( \mathcal{F}^X \)-measurable and weakly \( \mathcal{M}^{\text{dyad}}_{(s_l, t_l]} \)-invariant random variables \( Z_1, \ldots, Z_N : \Omega \to \mathbb{R} \) for \( l \in \{1, \ldots, L\} \), and let \( f : \mathbb{R}^N \to \mathbb{R} \) be a Borel function with \( F = f(Z_1, \ldots, Z_N) \in L^2(\mathcal{F}^X) \). Then, there are \( (\mathcal{B}((0, 1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n} \)-measurable and symmetric chaos kernels \( \tilde{f}_n \) for \( F \). In particular, they are constant on the cuboids

\[ \prod_{j=1}^n (s_{l_j}, t_{l_j}] \text{ for } l_1, \ldots, l_n \in \{1, \ldots, L\}. \]

**Proof.** The random variables \( \varphi(Z_k), k = 1, \ldots, N, \) are \( \mathcal{M}^{\text{dyad}}_{(s_l, t_l]} \)-invariant, where \( \varphi(x) := \arctan(x) \). Letting \( \psi(y) := \tan(y) \) for \( y \in (-\pi/2, \pi/2) \) and \( \psi(y) := 0 \) otherwise, and using the change of variables \( g(y_1, \ldots, y_N) := f(\psi(y_1), \ldots, \psi(y_N)) \), Lemma 4.3.4 implies that \( F = g(\varphi(Z_1), \ldots, \varphi(Z_N)) \) is \( \mathcal{M}^{\text{dyad}}_{(s_l, t_l]} \)-invariant. The sets \( E_l := (s_l, t_l] \) if \( t_l \) is not dyadic and \( E_l := (s_l, t_l) \) otherwise belong to \( \mathcal{O}((0, 1]) \). According to Lemma 4.5.1 the set \( E_l \) is locally ergodic with respect to \( \mathcal{M}^{\text{dyad}}_{E_l} \) and therefore with respect to the group generated by \( \mathcal{M}^{\text{dyad}}_{E_1}, \ldots, \mathcal{M}^{\text{dyad}}_{E_L} \). Furthermore, observing that \( \mathcal{M}^{\text{dyad}}_{E_l} = \mathcal{M}^{\text{dyad}}_{(s_l, t_l]} = \mathcal{M}^{\text{dyad}}_{(s_l, t_l)} \) if \( t_l \) is not dyadic, Theorem 4.5.3 gives the existence of symmetric kernels \( f_n \) that are \( (\mathcal{B}((0, 1])_E \otimes \mathcal{B}(\mathbb{R}))^{\otimes n} \)-measurable. Modifying the kernels on a null set yields the assertion. \( \square \)

**Doléans-Dade stochastic exponential**

We follow [GGL13] and assume \( X \) to be \( L^2 \)-integrable and of mean zero. For \( 0 \leq a \leq t \leq 1 \) we let

\[ S_t^a := 1 + \sum_{n=1}^\infty \frac{I_n((a, t])^{\otimes n}}{n!}, \]

where we can assume that all paths of \( (S_t^a)_{t \in [a, 1]} \) are càdlàg for any fixed \( a \in [0, 1] \). Then we get that

\[ S_t^a = 1 + \int_{[a, t]} S_u^a - dX_u \text{ a.s. and } S_t = S_t^0 S_a \text{ a.s. with } S_t := S_t^0. \]
Therefore we get from the chaos representation of $S^a_t$:

**Lemma 4.6.4.** Each random variable $S^a_{t_k}$ is $M^{\text{dyad}}_{[s_k, t_k]}$-invariant for $k = 1, \ldots, L$.

One could continue the investigation by using more general Doléans-Dade exponential formulas (see for example [Pro04, Chapter II, Theorem 37]), which is not done here.

**Limit functionals**

Behind the next examples there is common idea formulated in

**Definition 4.6.5.** For $0 \leq s < t \leq 1$ a random variable $Z: \Omega \to \mathbb{R}$ belongs to the class $C(s, t]$ provided that there exists a sequence $0 \leq N_1 < N_2 < \ldots$ of integers and Borel functions $\Phi_k: \mathbb{R}^{M_k} \to \mathbb{R}$ such that

$$Z = \lim_{k \to \infty} Z^k := \lim_{k \to \infty} \Phi_k \left( X_{a_k} - X_s, X_{a_k + 1} - X_s, \ldots, X_{b_k - 1} - X_s, X_t - X_{a_k + 2} \right)$$

a.s.,

where $\frac{a_k}{2^{N_k}}$ is the smallest grid point greater than or equal to $s$ and $\frac{b_k}{2^{N_k}}$ is the largest grid point smaller than or equal to $t$, $M_k := b_k - a_k + 2$, and the function $\Phi_k$ is symmetric in its arguments where the first and last coordinate are excluded.

**Proposition 4.6.6.** Let $Z_1, \ldots, Z_L: \Omega \to \mathbb{R}$ be random variables such that $Z_l$ belongs to the class $C(s_l, t_l]$ for $l = 1, \ldots, L$, and let $f: \mathbb{R}^L \to \mathbb{R}$ be a Borel function with $F := f(Z_1, \ldots, Z_L) \in L^2(\mathbb{F}^X)$. Then, there are $(B([0, 1])_E \otimes B(\mathbb{R}))^n$-measurable and symmetric chaos kernels $f_n$ for $F$. In particular, they are constant on the cuboids

$$\prod_{j=1}^n (s_{l_j}, t_{l_j}] \text{ for } l_1, \ldots, l_n \in \{1, \ldots, L\}.$$
Definition 4.6.5, there is an approximation $Z_m = \lim_{k \to \infty} Z^k_m$ a.s. By construction, there is a $k_0 \geq 1$ such that for all $k \geq k_0$ one has that $\varphi(Z^k_m)$ is $T_\beta$-invariant (here one has to distinguish between the cases $m = l$ and $m \neq l$). By dominated convergence, $\lim_{k \to \infty} \varphi(Z^k_m) = \varphi(Z_m)$ in $L^2(F^X)$ so that $\varphi(Z_m)$ is invariant with respect to $T_\beta$ as well and the proof is complete.

**Example 4.6.7.** For $0 \leq s < t \leq 1$ the following random variables belong to the class $C(s, t)$:

1. $X_t - X_s$
2. $[X, X]_t - [X, X]_s$, where $[X, X]$ is the quadratic variation process of $X$.
3. $\sup_{r \in (s, t]} |X_r - X_{r-}|$.

**Proof.** (1) is obvious. (2): Here we first take $\Phi_k(x_1, \ldots, x_{M_k}) := |x_1|^2 + \cdots + |x_{M_k}|^2$ with $N_k = k \geq k_0$, use [Pro04, Chapter II, Theorem 22] to get a sequence that converges in probability, and extract a sub-sequence that converges almost surely.

(3) Taking $\Phi_k(x_1, \ldots, x_{M_k}) := \max\{|x_1|, \ldots, |x_{M_k}|\}$ and $N_k := k$ with $k \geq k_0$ and the uniformity result for càdlàg paths [Bil99, Chapter 3, Lemma 1] yield the assertion.

**Remark 4.6.8.** Combining Proposition 4.6.6 with Example 4.6.7 (1) yields that the symmetric chaos kernels $f_n$ of $F = f(X_t - X_s, \ldots, X_{t_L} - X_{s_L})$ can be chosen to be constant on the cuboids

$$\prod_{j=1}^n (s_j, t_j) \quad \text{for} \quad t_1, \ldots, t_n \in \{1, \ldots, L\}.$$  

This was used in [GS13] in the investigation of variational properties of backward stochastic differential equations driven by Lévy processes. We explain the situation in the following example.

**Example 4.6.9 (An application to BSDEs).** Consider a backward stochastic differential equation

$$Y_t = F + \int_t^1 f(s, Y_s, Z_s, U_{s, \cdot})\, ds - \int_t^1 Z_s \, dB_s - \int_t^1 \int_{\mathbb{R}\{0\}} U_{s, x} \, d\tilde{N}_{s, x},$$

where $\tilde{N}$ is the compensated Poisson random measure associated to $X$, $B$ is the normalized Brownian part of the Lévy process $X$ and $f$ is an appropriate deterministic generator (for the precise setting see [GS13]). Given the initial data $F \in L^2(F^X)$ and $f$, one looks for the solution processes $(Y_t)_{t \in [0, 1]}$, $(Z_t)_{t \in [0, 1]}$ and $(U_{t, x})_{(t, x) \in [0, 1] \times (\mathbb{R}\{0\})}$.

To be able to control the variations

$$\|Y_t - Y_s\|_2 \quad \text{and} \quad \|Z_t - Z_s\|_2,$$

we have

$$\|Y_t - Y_s\|_2 \quad \text{and} \quad \|Z_t - Z_s\|_2,$$  

(4.6.1)
the authors in [GS13] assume a time net \(0 = r_1 < \cdots < r_L = 1\) such that the kernels \(f_n\) in the chaos expansion \(F = \sum_{n=0}^{\infty} I_n(f_n)\) of the terminal condition are constant on all cuboids

\[Q_{l_1,\ldots,l_n} := (r_{l_1-1}, r_{l_1}] \times \cdots \times (r_{l_n-1}, r_{l_n}].\]

One main step in [GS13] consists in verifying that the structure of the terminal condition transfers to the solution processes \(Y, Z\) and \(U\), i.e. \(Y_t, Z_t\) and \(U_{t,x}\) have chaos kernels that are constant on \(Q_{l_1,\ldots,l_n} \cap (0,t)^{\otimes n}\) and zero outside \((0,t)^{\otimes n}\). This is done by a Picard iteration, where (for example) the term \(\int_0^t f(s, Y_s, Z_s, U_s, \cdot) \, ds\) has to be handled within this iteration. Choosing sets of type \(E_{r_l} := (r_{l-1} \wedge r, r_l \wedge r]\), this can be done by Theorem 4.5.3. The knowledge of the \(L^2\)-variations in (4.6.1) is the key for the design of approximation and simulation schemes for BSDEs. In the particular case that \(X\) is the Brownian motion this allows probabilistic methods to solve semi-linear parabolic PDEs.

4.7 Summary and outlook

We were able to find a class of subspaces of \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) that can be represented by multiple integrals defined on suitable kernel spaces. If the invariances are locally ergodic, then this yields a Fock space structure, i.e., the \(n\)-th kernel space is representable as \(n\)-fold symmetric tensor product of the first kernel space.

Many random variables in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) occurring in applications possess invariances with respect to time permutations of the underlying Lévy process. This results in a simpler chaos expansion. By the considerations of the last section, the invariance properties of random variables in \(L^0(\Omega, \mathcal{F}, \mathbb{P})\) with no second moments can be described by the \(L^2\)-operators \(T_g\) as well.

This part and the paper [BG14] are supposed to motivate the study of properties of Lévy random variables by means of the chaos expansion. Time permutation of the underlying Lévy process is just one of the possible invariances. If one has chaos expansions for multidimensional Lévy processes in mind (cf. Part III) also spatial invariances like rotational symmetries, mirroring or self-similarities could serve as starting points for a similar methodology in these situations.
II

INFINITE DIMENSIONAL LÉVY PROCESSES
Motivated by the very general settings of [Nua95] and [LP11] for chaos expansions in the Gaussian and Poisson case, respectively, the question arose, to which extent one could generalize the chaos expansion for Lévy processes. By a generalization we have the state space of the Lévy process in mind. As it turns out, a Lévy-Itô decomposition is essential for the proof of the chaos expansion for Lévy processes, cf. Part III. In this part, we will therefore concentrate on obtaining such a result. But not only for the application to chaos expansions this is of interest. Considering Lévy processes on some state space, it is desirable to have as much insight in the structure of the process as possible. The Lévy-Itô decomposition is the main theorem one has to prove considering new situations. In the following, we give an overview about known results, state spaces under consideration in the past and present, about the situation which is investigated in this thesis and about core ideas how to treat infinitely divisible measures and Lévy processes in this case.

**State spaces of Lévy processes.**

Slight adaptations of the definition of Lévy processes allow a much more general state space than \( \mathbb{R}^d \). Several generalizations are imaginable and have been considered:

- **Banach spaces:** In recent research, Lévy processes with values in separable Banach spaces have been investigated e.g. in [AR02, AR05, ZR06, App07, RvG09] and earlier, by Dettweiler [Det83]. In the latter paper, the Lévy-Itô decompo-
5 Introduction

sition for Banach spaces was first proved, with a rather sketchy proof but still containing essential ideas which we also need in this thesis. The later papers by Albeverio and Rüdiger [AR02, AR05] pose more restrictive assumptions (type-2 Banach space) but obtain a strong integral version for the small jumps part in exchange. Also the definition of a stochastic integral is intimately connected to the geometry of the Banach space, cf. [Det83, MR09]. The proof of Riedle and van Gaans in [RvG09, Section 6] is essentially a reformulation of [Det83], but only one application of their Pettis integral definition which is an approach to get rid of these restrictions.

The main reason for considering Banach space valued Lévy processes are stochastic abstract Cauchy problems where the appearing function spaces are separable Banach spaces. Under various conditions (type, cotype, UMD, . . .) one obtains mild solutions with the help of a variation-of-parameters formula (cf. [AMR09]). In the case of SPDEs with Lévy noise in Hilbert spaces, the textbook [PZ07] of Peszat and Zabczyk covers the theory as well as many relevant examples.

Other topological vector spaces: Relaxing assumptions on the state space, one can consider separable Fréchet (complete locally convex metric) spaces. But in fact, every finite measure will be concentrated on a separable and reflexive Banach subspace, cf. [Bog91, Theorem 3.6.5], so that this is not really a more general situation. A generalization can be achieved though, if there are uncountably many seminorms generating the topology of a locally convex space $E$. In the special case of Suslin nuclear duals of nuclear spaces, Üstünel developed a stochastic integral [Üst82] and proved a kind of Lévy-Itô decomposition [Üst84]. The latter has the drawback of a small-jumps expression converging only in an $L^2$-sense; this and the lack of an independence assertion are the big difference to a desirable pathwise decomposition of the given Lévy process into independent processes.

Recently, C. Fonseca Mora [FM15] has considered Lévy processes with values in the dual of a reflexive nuclear space (but without any Suslin assumption – in contrast to Üstünel’s constructions) and proved a Lévy-Itô decomposition without the mentioned shortcomings.

Apart from Lévy processes, many authors have considered SPDEs in nuclear spaces or duals of nuclear spaces [Üst82, BG86, KPA88, KPA89, BJ91, KX95a, KX95b, GL98] with emphasis on distribution-valued processes and (semi)martingales, cf. the monograph of Itô [Itô84] or [Wul78, Itô80]. In particular, $S'$-valued processes are often motivated by applications e.g. in branching process theory [BGT07, GNR05, Kum11] or queuing theory [Ta109, TR09].

Rather general locally convex state spaces for Wiener processes are treated in [Bog91, Section 7.2] and [FdLP95].
Locally compact and Lie groups: Important results on infinitely divisible measures and convolution semigroups on locally compact groups like Lévy-Khintchine-type formulae and embeddability theorems have been obtained by Hunt [Hun56] and Siebert [Sie74]. A comprehensive theory of infinitely divisible measures can be found in the book of Heyer [Hey10]. Very early, Itô considered Brownian motion on Lie groups, cf. [Itô50]. The recent monograph of Applebaum [App14] covers the main theory in this research field which will not be of our concern in this thesis, though.

Cylindrical processes: The concept of a stochastic process in a Banach space $E$ is generalized in the sense that $X_t$ is not a classical random variable any more; it is a linear mapping from the topological dual space $E'$ to $L^0(\Omega, \mathcal{F}, P)$ in the spirit of the theory of cylindrical measures, cf. [Sch73, Part II]. Cylindrical Lévy processes can serve as suitable noise process in SPDEs, see [AR10, Rie11] with the help of so-called radonifying operators for the stochastic integral. Considering the definitions and theorems for cylindrical measures in [Sch73] including many topological vector spaces, generalizations to more general topological vector spaces seem possible also in this case.

Research focus.

In this thesis we consider a very general class of state spaces: Complete locally convex Suslin spaces.

It should be pointed out that not every separable locally convex space is Suslin. A counterexample is given in [CRdF00, Section 2.1]: The space of continuous functions on the countable product space of $\mathbb{N}$ provided with the topology of uniform convergence on compact sets of $\mathbb{N}^\mathbb{N}$ is separable and locally convex but not Suslin.

As mentioned, the special case of Wiener processes is studied in the comprehensive monograph of Bogachev, cf. [Bog91], in [FdLP95] and others. Although we make a sidestep to this topic in Section 7.7, the main concern in the following will be the jump part and especially its peculiarities with infinite activity. As a starting point, we suggest an approach which takes into account some ideas of Dettweiler (cf. [Det76a, Det83]) in order to construct a compensated Poisson integral and we prove a decomposition of sample paths, a Lévy-Itô decomposition. This extends the results of [Üst84] and [Det83] in the special cases mentioned above. We will instead use a path-wise definition of the integral defining the small jumps of the Lévy process by using a suitable Banach subspace. Also a weaker metric space will be used in the proofs. Let us give a short summary on these methods.
Comparing techniques.

The following comparing techniques are the most important tools in order to handle the uncountable neighbourhood bases of locally convex spaces beyond Fréchet spaces. We use close relations of the locally convex Suslin space \((E, \tau)\) to several metrizable spaces with comparable Suslin topologies. There are three main types of these spaces related to \(E\).

- **The abstract Polish space.** A Suslin space is a continuous image of some Polish space by definition. However, this space is rarely explicitly known and one cannot assume a locally convex structure on it. Therefore, this is merely a theoretical fact and hardly of practical use.

- **A weaker metric space.** In a completely regular Suslin space \(E\), every family of continuous functions separating the points of \(E\) has a countable subfamily, cf. [CV77, Lemma III.31]. If in addition, \(E\) is locally convex, this implies the existence of a sequence \((a_n)_{n \in \mathbb{N}}\) of elements in \(E'\) separating the points of \(E\). The map

\[
d(x, y) := \sum_{n=1}^{\infty} 2^{-n} |\langle x, a_n \rangle - \langle y, a_n \rangle|, \quad x, y \in E,
\]

is a locally convex metric on \(E\) which generates a weaker topology than the original one. We say \(d\) is a weaker metric on \(E\). As \((E, d)\) carries a comparable Suslin topology to \(\tau\), the respective Borel sets on \(E\) are the same (cf. Proposition 6.1.2) which makes this a very important tool in order to prove measurability issues by means of countable neighbourhoods in the metric case. Mostly, the existence of a limit is obtained via the complete topology \(\tau\) and measurability can be proved with help of the possibly non-complete metric topology of \(d\) (e.g. in lemmas 7.5.4, 7.5.5 and 9.1.2). The idea of using the weaker metric space in the context of completely regular Suslin spaces was used for example in [Bal89] for different purposes than ours.

Locally convex Suslin spaces are examples of so-called submetric spaces, i.e., spaces with a continuous metric, by the above construction. This notion was introduced by A. Jakubowski for proving weak limit theorems in situations beyond Polish spaces, cf. [Jak97, Jak00]. Also in above mentioned paper of Castaing and Reynaud de Fitte [CRdF00], the existence of a sequence of continuous functions separating the points of \(E\) (and therefore a continuous metric) was assumed in the context of separable locally convex spaces in order to prove a strong law of large numbers for random elements in \(E\).

- **A compactly embedded Banach space.** In a locally convex space \(E\) one can consider so-called local normed spaces \(E_A\) which are linear hulls of absolutely
convex and bounded sets $A$ endowed with the Minkowski functional $\| \cdot \|_A$ of this set as a norm, cf. [Jar81, Section 8.3]. If this set is compact, one always obtains a Banach space which is continuously (even compactly) embedded in the original space $E$. If this space is additionally separable and $E$ is a Suslin space, the Borel $\sigma$-algebra induced by the norm topology and the subspace topology coincide (see Proposition 6.1.2). Such spaces will serve as auxiliary spaces for the convergence of an integral describing the small jump behaviour of a Lévy process with values in a locally convex Suslin space.

The starting point for the analysis of the small jumps is the Lévy-Khintchine decomposition (Theorem 6.2.5 as obtained in [Det76a]). For the description of the analogon of the small jump part, an absolutely convex compact disk $K$ handles the singular behaviour of the Lévy measure in some neighbourhood of zero. Starting from there, we attempt to let the small jumps live in the Banach space $E_K$. If the Lévy measure can be restricted to $E_K$ (which, as it turns out, is always possible in separably extendable spaces, cf. Definition 7.3.4), this leads to a suitable embedding $E_K \hookrightarrow E$ with all desired properties. The right reduction procedure is mainly investigated in Section 7.3.

**Outline**

This part is organized as follows. Chapter 6 presents notions and results in the areas of locally convex Suslin spaces, Radon measures and infinitely divisible measures on these spaces as well as the Lévy-Khintchine decomposition for infinitely divisible measures on complete locally convex spaces. Chapter 7 contains the main part. The definition of a Lévy process and its existence are presented in sections 7.1 and 7.2. In Section 7.3 we investigate the reduction of Lévy measures to separable compactly embedded Banach subspaces and in Section 7.4 we study the question, in which spaces Lévy measures can always be reduced. Section 7.5 provides some auxiliary results on càdlàg functions in Suslin spaces. In Section 7.7 we introduce our notion of a Wiener process and prove some series expansions needed in Part III. In Section 7.8 we obtain the main result of this part, the Lévy-Itô decomposition, Theorem 7.8.1.
6.1 Locally convex Suslin spaces

**Assumption 6.1.1.** In this part, we will make the following assumption on the state space $E$ (unless explicitly stated differently):

(S1) $E$ is a real locally convex space,

(S2) $E$ is complete, and

(S3) $E$ is a Suslin space.

A complete locally convex space has the property that every Cauchy net has a limit. A **Suslin space** is a Hausdorff (topological) space which is a surjective continuous image of a polish space. We denote by $E'$ the topological dual of $E$ and $\mathcal{B}(E)$ is the Borel-$\sigma$-algebra of $E$. The cylindrical $\sigma$-algebra $\mathcal{E}(E)$ is given by

$$\sigma\left\{ x \in E : (\langle x, a_1 \rangle, \ldots, \langle x, a_n \rangle) \in B \right\}, \ n \in \mathbb{N}, \ B \in \mathcal{B}(\mathbb{R}^n), \ a_1, \ldots, a_n \in E' .$$

The above assumptions guarantee the following frequently used properties:

(P1) $E$ is Hausdorff and completely regular (i.e. a point and a closed set can be separated by a continuous function), cf. [Bog91, p. 363].
(P2) \( E \) is separable (follows directly from the Suslin property).

(P3) \( \mathcal{E}(E) = \mathcal{B}(E) \), cf. [Sch73, Lemma 18, p. 108].

(P4) \((E, \mathcal{B}(E))\) is a measurable vector space, i.e., addition and scalar multiplication are measurable, cf. [VTC87, Proposition 2.3, p. 16].

The following essential result is due to L. Schwartz:

**Proposition 6.1.2** ([Sch73, Corollary 2, p. 101]). Let \( \tau_1 \) and \( \tau_2 \) two comparable Suslin topologies on a set \( F \). Then, the respective Borel-\(\sigma\)-algebras coincide.

### 6.2 Measures on locally convex Suslin spaces

#### 6.2.1 Preliminaries on measures.

**Definition 6.2.1** (Radon measures).

(1) The set of (nonnegative) Borel measures on \( E \) is denoted by \( \mathcal{M}(E) \), its subset of finite, resp. probability measures by \( \mathcal{M}^1(E) \), \( \mathcal{M}^\sigma(E) \), respectively.

(2) A measure \( \mu \in \mathcal{M}^\sigma(E) \) is called Radon, if for \( B \in \mathcal{B}(E) \) and \( \varepsilon > 0 \) there exists a compact set \( K \subseteq B \) such that \( \mu(B \setminus K) < \varepsilon \).

Every finite Borel measure on any Suslin space is Radon [Sch73, Thm. 10, p. 122]. The Fourier transform of a measure \( \mu \in \mathcal{M}^\sigma(E) \) is given by

\[
\hat{\mu} : E' \to \mathbb{C} \quad \hat{\mu}(\alpha) := \int_E \psi(x,\alpha) \, d\mu(x).
\]

Any two finite measures with equal Fourier transform coincide, cf. [VTC87, Theorem 2.2, p. 200]. For two measures \( \mu \) and \( \nu \) on \( \mathcal{B}(E) \) the convolution is defined by

\[
\mu * \nu(B) = (\mu \otimes \nu)_\alpha(B) = \mu \otimes \nu(\alpha^{-1}(B)),
\]

where \( \alpha : E \times E \to E, (x,y) \mapsto x + y \), cf. [Bog91, Appendix, p. 373f].

The notion of a Radon measure is not consistent in the literature. Occasionally, it is a term for a locally finite inner regular measure, e.g. in the monograph of Schwartz [Sch73]. In this paper we will either work with finite measures or with measures which are not even locally finite (some Lévy measures). Therefore, the given definition of a Radon measure will prove most useful.
6.2 Measures on locally convex Suslin spaces

**Definition 6.2.2 (Restriction).** For $A \in \mathcal{B}(E)$ the restrictions $\mu|_A \in \mathcal{M}(E)$ resp. $\mu\|_A \in \mathcal{M}(A)$ of a measure $\mu \in \mathcal{M}(E)$ to $A$ are defined by

$$
\mu|_A : \mathcal{B}(E) \to [0, \infty], \quad B \mapsto \mu(B \cap A) \text{ resp. } \\
\mu\|_A : \mathcal{B}(A) \to [0, \infty], \quad B \mapsto \mu(B \cap A).
$$

**Definition 6.2.3 (Weak convergence).** A net of measures $M \subseteq \mathcal{M}^b(E, \tau)$ is said to converge weakly iff the net $\{\mu(f) : \mu \in M\}$ converges in $\mathbb{R}$ for every $f \in \mathcal{C}^b(E, \tau)$. The topology of weak convergence is also called weak topology on $\mathcal{M}^b(E)$.

### 6.2.2 Infinitely divisible measures.

A measure $\mu \in \mathcal{M}^1(E)$ is called Poissonian provided that there exists $\nu \in \mathcal{M}^b(E)$ such that its Fourier transform satisfies

$$
\widehat{\mu}(a) = e^{\int_E (e^{i \langle x, a \rangle} - 1) \, d\nu(x)} \quad \text{for all } a \in E'.
$$

For a measure $\nu \in \mathcal{M}^b(E)$ its Poisson exponential can be defined by

$$
e(\nu) := e^{-\nu(E)} \sum_{n=0}^{\infty} \frac{\nu^n}{n!}, \quad (6.2.1)
$$

with a setwise converging series. In this case, $e(\nu)$ is indeed a Poissonian measure with associated measure $\nu$, cf. [Det76a, p. 288].

A measure $\varrho \in \mathcal{M}^1(E)$ is called Gaussian if for any $a \in E'$ the measure $\varrho \circ a^{-1}$ is Gaussian. Gaussian measures on locally convex spaces have been extensively studied by various authors, and the interested reader should be referred to the monograph [Bog91]. A measure $\varrho \in \mathcal{M}^1(E)$ is Gaussian if and only if its Fourier transform equals

$$
\widehat{\varrho}(a) := \exp \left( i \langle m, a \rangle - \frac{1}{2} \langle Qa, a \rangle \right),
$$

for all $a \in E'$, where $m \in E$ and the covariance operator $Q : E' \to E$ is a symmetric continuous ($E'$ carries the Mackey topology) and linear operator, cf. [Bog91, Section 3.2].

A measure $\mu \in \mathcal{M}^1(E)$ is called infinitely divisible provided that for every $n \in \mathbb{N}$ there exists a measure $\mu_n \in \mathcal{M}^1(E)$ such that $\mu = \mu_n^n$. Gaussian and Poisson measures are infinitely divisible. The set $I(E)$ of infinitely divisible measures on $E$ is closed in $\mathcal{M}^1(E)$ in the weak topology, cf. [Det76a, Korollar 1.10].

A set $M \subseteq \mathcal{M}^b(E)$ of finite measures on $E$ is uniformly tight if $\sup_{\mu \in M} \mu(E) < \infty$ and if for all $\varepsilon > 0$ there exists a compact set $K \subseteq E$ such that $\mu(K^c) < \varepsilon$ for all $\mu \in M$.
The set $M$ is called *shift tight*, if for every $\mu \in M$ there exists an $x_\mu \in E$ such that the family $(\mu * \delta_{x_\mu})_{\mu \in M}$ is uniformly tight.

**Definition 6.2.4.** A measure $\nu \in \mathcal{M}(E)$ is called a *Lévy measure* if it satisfies

1. $\nu(\{0\}) = 0$;
2. there exists an upwards directed set of finite measures (w.r.t. $\leq$) $\{\mu_i \in \mathcal{M}(E) : i \in I\}$ with $\mu_i \leq \nu$ and $\sup_i \mu_i = \nu$ (setwise) and such that the family of Poisson measures $(e(\mu_i))_{i \in I}$ is shift tight.

This definition is due to Dettweiler, cf. [Det76a]. If $E$ is a separable Banach space, this definition coincides with the following characterization of a Lévy measure:

- $\nu(\{0\}) = 0$,
- $\nu(B_\delta^c) < \infty$ for all $\delta > 0$ and where $B_\delta$ is the ball with radius $\delta$, and for each positive sequence $\delta_n \searrow 0$, the set $\{e(\nu|_{B_{\delta_n}^c}), n \in \mathbb{N}\}$ is shift tight, cf. [Hey10, Theorem 3.4.9].

The above definition is thus indeed an extension of well-known concepts.

There exists a Lévy-Khintchine decomposition of infinitely divisible measures on $E$:

**Theorem 6.2.5** (Dettweiler, [Det76a, Satz 2.5]). For $\mu \in \mathcal{I}(E)$ there exist $\gamma \in E$, a linear symmetric and positive definite operator $Q : E' \to E$, an absolutely convex and compact set $K \subseteq E$ and a Lévy measure $\nu$ such that the characteristic function of $\mu$ has the form

$$\tilde{\mu}(a) = \exp \left( i\langle \gamma, a \rangle - \frac{1}{2} \langle Qa, a \rangle + \int_E \left( e^{i\langle x, a \rangle} - 1 - i\langle x, a \rangle \mathbb{1}_K(x) \right) d\nu(x) \right)$$

for every $a \in E'$. $\nu$ and $Q$ are uniquely determined by $\mu$, $\gamma$ is unique after the choice of $K$.

Conversely, every measure $\mu \in \mathcal{M}^1(E)$ with a Fourier transform of this type is infinitely divisible.

One says that $\mu$ has *characteristics* $(\gamma, Q, \nu, K)$ if $\mu$ admits the above decomposition. The covariance operator is symmetric in the sense that $\langle Qa, b \rangle = \langle Qb, a \rangle$ for $a, b \in E'$.

We will frequently use the following useful corollary of the Lévy-Khintchine decomposition.

**Corollary 6.2.6.** For a measure $\nu \in \mathcal{M}(E)$ the following assertions are equivalent:

1. $\nu$ is a Lévy measure on $E$.
2. There exists a measure $\varepsilon \in \mathcal{M}^1(E)$ such that its Fourier transform equals

$$\int_E e^{i\langle x, a \rangle} d\varepsilon(x) = \exp \left( \int_E \left( e^{i\langle x, a \rangle} - 1 - i\langle x, a \rangle \mathbb{1}_K(x) \right) d\nu(x) \right), \quad a \in E'$$

for some absolutely convex and compact set $K \subseteq E$. 

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6.2 Measures on locally convex Suslin spaces

Proof. (2) ⇒ (1): If $\xi$ has the above form, $\xi \in \mathcal{I}(E)$ and $\nu$ is the unique Lévy measure corresponding to $\xi$ by Theorem 6.2.5.

(1) ⇒ (2): If $\nu$ is a Lévy measure, it follows from the proof of the Lévy-Khintchine decomposition that there exists a generalized Poisson exponential $\varepsilon$ with the given Fourier transform (end of point 1 in the proof of Satz 2.5, [Det76a], where this measure is called $\nu_0 \ast e(F_0)$). □
Recall that \( E \) is a complete locally convex Suslin space throughout this chapter. We start with the definition of a Lévy process with values in \( E \).

### 7.1 Lévy processes in locally convex spaces

An \( E \)-valued random vector \( X \) on a probability space \((\Omega, \mathcal{F}, P)\) is a measurable map \( X : (\Omega, \mathcal{F}) \to (E, \mathcal{B}(E)) \). An \( E \)-valued stochastic process \((X_t)_{t \in T}\) is a collection of \( E \)-valued random vectors over the parameter space \( T = [0, t_{\text{max}}]\) with \( t_{\text{max}} > 0 \) or \( T = [0, \infty) \). If there is no risk of confusion, we will omit the emphasis that a process is \( E \)-valued and call it simply a stochastic process.

**Definition 7.1.1 (Convolution semigroups).** A family \((\mu_t)_{t \in T}\) of probability measures on \((E, \mathcal{B}(E))\) is called a convolution semigroup, if \( \mu_{t+s} = \mu_t \ast \mu_s \) for all \( s, t, s + t \in T \) and \( \mu_0 = \delta_0 \). It is weakly continuous, if \( \mu_t \) converges weakly to \( \delta_0 \) for \( t \searrow 0 \), i.e. \( \mu_t(f) \to \delta_0(f) \) for every real-valued bounded and continuous function \( f \).

**Definition 7.1.2 (Lévy processes).** An \( E \)-valued stochastic process \((X_t)_{t \in T}\) on a probability space \((\Omega, \mathcal{F}, P)\) with distributions \( \mu_t := P_{X_t} \) is a Lévy process, if

(L1) \( X_t - X_s \) is independent of \( \mathcal{F}_s := \sigma(X_r : r \leq s) \) for any \( 0 \leq s < t \);
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(L2) given \( s \in T \), the distributions of \( X_{t+s} - X_t \) and \( X_s \) are equal for all \( t \in T \);

(L3) \( X_0 = 0 \) a.s.; and

(L4) the family \((\mu_t)_{t \in T}\) is a weakly continuous convolution semigroup.

If \( E = \mathbb{R}^d \), the above definition yields the notion of a Lévy process in law, cf. Definition 2.2.4, where (L4) is substituted by the equivalent property (cf. [App09, Proposition 1.4.1]) of stochastic continuity.

The following result was obtained by E. Siebert.

**Lemma 7.1.3** (Siebert [Sie74, Satz 6.4]). Let \( \mu \in \mathcal{I}(E) \). Then the set of roots

\[
W(\mu) = \bigcup_{n \in \mathbb{N}} \{ \nu^m : \nu \in \mathcal{M}^1(E), \nu^m = \mu, 1 \leq m \leq n \}
\]

is uniformly tight.

**Proposition 7.1.4.** Let \( \mu \in \mathcal{I}(E) \).

1. The \( n \)-th root \( \mu^{1/n} \in \mathcal{M}^1(E) \) is unique.

2. Let \( \eta : E' \to \mathbb{C} \) be the characteristic exponent of \( \mu \) in Theorem 6.25. Then, \( \hat{\mu}_t(a) = e^{\eta(a)}, a \in E', t \in T \).

3. There is a unique convolution semigroup \( \mu_t := \mu^{*t} \) embedded into \( \mu \) such that \( \mu_1 = \mu \).

**Proof.** (1): On a locally convex Suslin space, \( \mathcal{E}(E) = \mathcal{B}(E) \), therefore \( \mu \) is uniquely determined on the generating \( \pi \)-system of cylindrical sets of the form \( C = (a_1^{-1}, \ldots, a_d^{-1})(B) \) with \( B \in \mathcal{B}(\mathbb{R}^d) \) and \( a_1, \ldots, a_d \in E' \). So, let \( g_1 \) and \( g_2 \) be two different \( n \)-th roots of \( \mu \). Then, there exist \( a_1, \ldots, a_d \in E' \) such that \( g_1 \circ (a_1, \ldots, a_d)^{-1} \neq g_2 \circ (a_1, \ldots, a_d)^{-1} \). These are two different \( n \)-th roots of \( \mu \circ (a_1, \ldots, a_d)^{-1} \in \mathcal{I}(\mathbb{R}^d) \), a contradiction.

(2) and (3): From (1) it follows that \( \mu_q := \mu^{*q} \) is unique and definable for all rational \( q \) by setting \( \mu^{*q}/q_2 := \mu^{1/q_1}_1 \). For \( a \in E' \), the measure \( \mu^{*q}_a := \mu^{*q} \circ a^{-1} \) is infinitely divisible, and from the one-dimensional case it follows that its Fourier transform is \( \varphi^q_{\mu}(u) = e^{\eta(q)(u)} = e^{\eta(q)(ua)} \), where \( \eta : E' \to \mathbb{C} \) is the characteristic exponent of \( \mu \), i.e. \( \hat{\mu}(a) = e^{\eta(a)} \). This follows from \( \nu_a \) being a Lévy measure on \( \mathbb{R} \) (cf. [Hey10, Theorem 3.4.9]). For \( q \searrow t \), the Fourier transforms \( \varphi^q_{\mu}(u) \to \varphi^t_{\mu}(u) \) for all \( u \in \mathbb{R} \), which yields (2). In other words, \( \mu^{*q}(a) \to \mu^{*t}(a) \) for all \( a \in E' \). The family \( (\mu^{*q})_{q \in \mathbb{Q}, t \leq q \leq t_0} \) is uniformly tight by Lemma 7.1.3 and therefore weakly relatively compact by Prokhorov’s theorem cf. [VTC87, Theorem I.3.6]. This is sufficient to apply [VTC87, Theorem IV.3.1] and obtain weak convergence of the net \( \mu^{*q}, q \in \mathbb{Q}, q \geq t \) for \( q \searrow t \). \( \square \)
We prove that there exists a process satisfying (L1)-(L4). This is obtained in Proposition 7.2.5 by extending finite-dimensional distributions on the path space $E^T$. Nevertheless, some of the needed facts are not standard for the general case, so we will carry out the construction in detail.\(^1\) On the way we need a result on uniform tightness of the convolution semigroup.

**Lemma 7.2.1.** Let $\Gamma \subseteq \mathcal{M}^1(E)$ be a uniformly tight family of probability measures on $E$. Then its weak closure $\overline{\Gamma}$ is uniformly tight.

**Proof.** A theorem of A. D. Alexandrov (cf. [VTC87, Theorem I.3.5]) establishes equivalence of weak convergence of a directed set of measures $(\mu_r)_{r<t}$ towards a measure $\mu_t$ and the condition $\overline{\lim} \mu_r(K) \leq \mu_t(K)$ for all closed sets $K$. For $\gamma \in \overline{\Gamma} \setminus \Gamma$ there exists a net $(\mu_r)_{r} \subseteq \Gamma$ with limit $\gamma = \lim_r \mu_r$. By definition, the net is uniformly tight. Then for $\varepsilon > 0$ there exists a compact (thus closed, as a Suslin space is Hausdorff) set $K$ such that $\mu_r(K) > 1 - \varepsilon$ for all $r$. This implies that $\gamma(K) \geq \overline{\lim} \mu_r(K) \geq 1 - \varepsilon$ and the assertion follows by taking $K_{\varepsilon/2}$ corresponding to $\varepsilon/2$ in order to satisfy the strict inequality. \(\square\)

**Corollary 7.2.2.** The set $\{\mu_t : 0 \leq t \leq t_0\}$ of distributions of a weakly continuous convolution semigroup in $E$ on a compact interval $T = [0, t_0]$ is uniformly tight.

**Proof.** We have that $\mu_{t_0}$ is infinitely divisible and $\mu_{t_0/n}$ is an $n$-th root. Lemma 7.1.3 yields uniform tightness of the rational roots $\mu_{rt_0}$, $r \in \mathbb{Q} \cap [0,1]$, of $\mu_{t_0}$. As $\mu_t = \lim_{r \searrow t, r \in \mathbb{Q}} \mu_r$, all $\mu_t$ with irrational index are in the closure of the set of roots. Therefore, the result follows by Lemma 7.2.1. \(\square\)

We are going to define the process by its Markov transition kernels by setting $P^{s,t,x} := \mu_{t-s}(\cdot - x)$ which satisfy the Chapman-Kolmogorov equations, (cf. [Sch77, Relation (3.1)]) due to the semigroup property. It remains to show the continuity conditions of Schwartz for the existence of an almost sure càdlàg process with values in Suslin spaces defined by the transition kernels, cf. [Sch77, Section 4].\(^2\) The following lemma shows that weakly continuous convolution semigroups are not only continuous from above.

**Lemma 7.2.3.** Let $(\mu_t)_{t \geq 0} \subseteq \mathcal{M}^1(E)$ be a weakly continuous semigroup. Then, $\mu_s$ converges weakly to $\mu_t$ for $s \to t$.

\(^1\)An alternative existence proof is obtained as a side result on the way to a Lévy-Itô decomposition, as a Lévy process with given distribution is being constructed in Section 7.6.

\(^2\)In sections 2 and 3 of [Sch77] the assumptions on the transition kernels are posed. In section 4, the probability measure on the path space is constructed tacitly using these conditions for the deductions.
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Proof. Proposition 7.1.4 yields pointwise convergence of \( \tilde{\mu}_s \) to \( \tilde{\mu}_t \) for \( s \to t \). Let \( s_\alpha, \alpha \in A \) be the converging net, it is bounded by some \( s_0 > 0 \). The family of distributions \( \mu_\alpha \) for \( 0 \leq s \leq s_0 \) is relatively compact by Corollary 7.2.2 and Prokhorov’s theorem, in particular \( \mu_\alpha \). This is sufficient to obtain weak convergence of the net \( \mu_\alpha \) to \( \mu_t \) by [VTC87, Theorem IV.3.1].

Lemma 7.2.4. Let \( (\mu_t)_t \) be a weakly continuous convolution semigroup on \( E \). For \( s < t \) and \( x \in E \), the transition kernels \( P^{s,t,x} := \mu_{t-s}(\cdot - x) \) are

1. normal, i.e. \( P^{s,t,y} = \delta_y \),

2. uniformly tight for \( (s,t,x) \in [0,s_0] \times [0,t_0] \times H \) for every compact set \( H \subseteq E \) and

3. weakly continuous on \( \mathbb{R}_+ \times \mathbb{R}_+ \times H \) for every compact set \( H \subseteq E \).

Proof. (1) is obvious. (2) follows from uniform tightness of the family \( \mu_t \) for finite time intervals. For, if \( \mu_t(K) > 1 - \varepsilon \) for all \( t \in [0,t_0] \) and a compact set \( K \), one has \( 1 - \varepsilon < \mu_t(K) \leq \mu_t(H - x) \) as \( H - x \) for \( x \in H \).

The map \( (s,t,x) \mapsto P^{s,t,x} \) is weakly continuous if for a net \( (s,t,x) \in [0,s_0] \times [0,t_0] \times H \) converging to \( (s',t',x') \) the net of measures \( (P^{s,t,x})_{\alpha} \) converges weakly to \( P^{s',t',x'} \). A sufficient condition for weak convergence is the pointwise convergence of the Fourier transforms if the net is weakly relatively compact, [VTC87, Theorem IV.3.1]. First, uniform tightness of the transition kernels implies relative compactness due to Prokhorov’s theorem, cf. [VTC87, Theorem I.3.6]. Note that \( (s,t,x) \in (s_\alpha, t_\alpha, x_\alpha) \) converges in each component. For \( a \in E' \) one obtains

\[ P^{s,t,x}(a) = \tilde{\mu}_{s-t}(a) \tilde{\delta}_{x_\alpha}(a) \to \tilde{\mu}_{s-t}(a) \tilde{\delta}_{x'}(a) = P^{s',t',x'}(a) \]

by Lemma 7.2.3 and continuity of the map \( x \mapsto \tilde{\delta}_x(a) = \delta^{(x,a)} \). Furthermore, it was used that the product of two Cauchy nets of numbers converges to the product of the limits. The assertion holds for all bounded time intervals, but if the map is continuous on all bounded intervals, then it is on \( \mathbb{R}_+ \times \mathbb{R}_+ \times H \).

Proposition 7.2.5. Let \( (\mu_t)_{t \in T} \) be a weakly continuous convolution semigroup on \( E \). Then, there exists a Lévy process \( (X_t)_{t \in T} \) with values in \( E \) and such that \( P_{X_t} = \mu_t \) for all \( t \in T \). Furthermore, one can choose a probability space such that the set of càdlàg paths has probability one.

Proof. We first show that there exists a Markov process according to the transition kernels and afterwards that it satisfies (L1) – (L4).

1. One can define the probability measure \( \mathbb{P} \) on the path space \( E^T \) endowed with the


\[ \mathbb{P}(X_{t_0} \in B_0, \ldots, X_{t_n} \in B_n) = \int \cdots \int_E 1_B(y_0 + \cdots + y_n) \, d\mu_{t_n-t_{n-1}}(y_n) \times \cdots \\
\cdots \times 1_B(y_0 + y_1) \, d\mu_{t_1-t_0}(y_1) 1_B(y_0) \, d\mu_{t_0}(y_0) \]

for \( 0 \leq t_0 < t_1 < \ldots < t_n < \infty \), \( t_k \in T \), \( B_k \in \mathcal{B}(E) \), \( k = 0, \ldots, n \) and \( n \in \mathbb{N} \). The properties of Lemma 7.2.4 and the Chapman-Kolmogorov relations are sufficient for applying Kolmogorov’s extension theorem, carried out in [Sch77, Section 4]. This yields a probability measure \( \mathbb{P} \) on the path space \( E^T \) and a Markov process with the given transition kernels. A further construction step, cf. [Sch77, Théorème (8.3)], yields a probability measure (which we again call \( \mathbb{P} \)) with the property that \( \mathbb{P}(D(T; E)) = 1 \).

\( D(T; E) \) denotes the set of càdlàg functions \( \xi : T \to E \).

(2) We show that the Markov process \( (X_t)_{t \in T} \) satisfies (L1) – (L4).

(L4): The existence of the weakly continuous semigroup has been assumed beforehand. From the above equality one immediately obtains \( \mathbb{P}(X_t = \mu_t) \).

(L3): One has \( \mathbb{P}(X_0 = 0) = \mu_0(\{0\}) = 1 \).

(L2): Stationary increments. For \( s < t \) one has

\[ \mathbb{P}(X_t - X_s \in B) = \mathbb{P}(\varphi(X_t, X_s) \in B) = \int \int_E \varphi(y_0 + y_1, y_0) \, d\mu_{t-s}(y_1) \, d\mu_s(y_0) = \int \int_E 1_B(y_1) \, d\mu_{t-s}(y_1) \, d\mu_s(y_0) = \mu_{t-s}(B) = \mathbb{P}(X_{t-s} \in B), \]

for any Borel set \( B \in \mathcal{B}(E) \) and \( \varphi(x, y) = x - y \) which is the assertion.

(L1) Independent increments. For Borel sets \( B_1, \ldots, B_n \) and \( \varphi \) as before let

\[ f(x_0, x_1, \ldots, x_n) = \prod_{k=1}^n 1_{B_k}(x_k - x_{k-1}) = \prod_{k=1}^n 1_{B_k}(\varphi(x_k, x_{k-1})). \]

Then, one obtains

\[ \mathbb{P}(X_t - X_{t_0} \in B_1, \ldots, X_{t_n} - X_{t_{n-1}} \in B_n) = \mathbb{E}(f(X_{t_0}, \ldots, X_{t_n})) = \int_E \cdots \int_E f(y_0, y_0 + y_1, \ldots, y_0 + \cdots + y_n) \, d\mu_{n-t_{n-1}}(y_n) \cdots d\mu_{t_1-t_0}(y_1) \, d\mu_{t_0}(y_0) \]

for any Borel set \( B \in \mathcal{B}(E) \) and \( \varphi(x, y) = x - y \) which is the assertion.
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\[
\int_E \cdots \int_{E_{k=1}}^n 1_{B_k}(y_k) \, d\mu_{t_n-t_{n-1}}(y_n) \cdots d\mu_{t_1-t_0}(y_1) \, d\mu_{t_0}(y_0)
\]

\[
= \prod_{k=1}^n \mathbb{P}(X_{t_k} - X_{t_{k-1}} \in B_k),
\]

where stationarity was used for the last equality.

**Assumption 7.2.6.** From now on, we assume that \(E\)-valued Lévy processes \(X = (X_t)_{t \in T}\) are always given on a complete probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that all paths of \(X\) are càdlàg (they are càdlàg on a set of measure one; one the complement we set all trajectories to zero).

For \(\omega \in \Omega\) define

\[
X_{t^-}(\omega) := \lim_{s \nearrow t} X_s(\omega) \quad \text{and} \quad \Delta X_t(\omega) := X_t(\omega) - X_{t^-}(\omega).
\]

### 7.3 Separable Banach subspaces

Our main strategy on the way to a decomposition of sample paths consists in storing the small jumps into a Banach subspace \(E_K\) of \(E\) and treating the big jumps as a finite sum. In other words, we split the Lévy measure into two parts, such that the distribution of the small jumps has a Banach support. This will require conditions on the space or the Lévy measure. For the functional analytic background of the following cf. \[Jar81\]. A set in a locally convex space is called bounded if it is absorbed (that means in some homothetic image) by every zero-neighbourhood. For a locally convex space \(E\) we define the system \(K_0 := K_0(E) := \{ K \subseteq E : K \text{ compact disk} \}\), where a disk is a non-empty bounded and absolutely convex set. For a disk \(K \subset E\) the linear hull

\[
E_K := \bigcup_{n \in \mathbb{N}} n \cdot K
\]

is a normed space with respect to the gauge function

\[
\|x\|_K := \inf\{ \rho > 0 : x \in \rho \cdot K \}, \quad x \in E_K.
\]

If \((E_K, \| \cdot \|_K)\) is complete, \(K\) is called a Banach disk. By boundedness of \(K\) in \(E\), the canonical injection \(i : E_K \hookrightarrow E\) is continuous (if \(K\) is compact, even a compact mapping), therefore the norm topology on \(E_K\) is finer as the induced topology. The continuity of the embedding together with \[Jar81, \text{Theorem 3.2.4}\] imply that \(K \in \mathcal{K}_0\) is always a Banach disk.
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**Lemma 7.3.1** (Dettweiler [Det76a], Lemma 1.5/Proof of Satz 2.5). Let $\mu \in \mathcal{I}(E)$. Then there exists a unique Lévy measure $\nu$ associated to $\mu$ and $K \in \mathcal{K}_0$ such that $\nu(K^c) < \infty$.

**Definition 7.3.2.** Let $\nu$ be a Lévy measure and $x_\mu \in E$ be chosen in a way such that the family $(e(\mu) * \delta_{x_\mu})_{\mu \leq \nu}$ is uniformly tight. The generalized Poisson exponential $\tilde{e}(\nu)$ is an accumulation point of this family.

The generalized Poisson exponential is unique up to translations [Det76a, p. 288]. We will sometimes use $\tilde{e}(\nu)$ in relations like

$$\tilde{e}(\nu) = \tilde{e}(\nu|_K) * e(\nu|_{K^c})$$

meaning that choosing certain representatives of the generalized Poisson exponential, the equality holds up to a convolution with a Dirac measure on one side.

By Prokhorov’s theorem, cf. [VTC87, Theorem I.3.6], the mentioned family is weakly relatively compact which justifies the definition as an accumulation point. We will use the fact that for a Lévy measure, its generalized Poisson exponential $\tilde{e}(\nu)$ is infinitely divisible. This follows from $\mathcal{I}(E)$ being closed in $\mathcal{M}^1(E)$.

### 7.3.1 Measures on different underlying topologies

We start with a lemma motivating the notion of local separability and conclude with lemmas on certain types of measures with respect to different topologies.

**Lemma 7.3.3.** Let $\tau$ denote the original topology on $E$ and its restrictions to subspaces.

(1) Let $K$ be a closed disk. The subspace $(E_K, \tau)$ is a Suslin space.

(2) Let $K$ be a closed disk such that $(E_K, \|\cdot\|_K)$ is a separable Banach space. Then,

$$\mathcal{E}(E_K, E_K^{'}) = \mathcal{B}(E_K, \|\cdot\|_K) = \mathcal{B}(E_K, \tau) = \mathcal{E}(E_K, E_K'|_{E_K})$$

where $E_K'$ denotes the linear and continuous functionals on $E_K$ with respect to the norm topology.

**Proof.** (1) follows from the fact that $K$ is closed, thus $E_K$ is Borel and every Borel subset of a Suslin space is Suslin [Sch73, Thm. 3, p. 96].

(2): $E_K$ is a Suslin space due to its separability, thus $\mathcal{B}(E_K) = \mathcal{E}(E_K)$, the same is true for the induced topology on $E_K$ which is Suslin by (1). Finally, two comparable Suslin topologies have the same Borel sets by Proposition 6.1.2. \qed

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**Definition 7.3.4** (Reducing notions). (1) Let $\nu \in \mathcal{M}(E)$ be a Lévy measure. $E$ is called $\nu$-locally separable if there exists a compact disk $K$ such that $\nu(K^c) < \infty$ and $E_K$ is a separable Banach space with the norm $\| \cdot \|_K$.

(2) $E$ is called locally separable provided that $E$ is $\nu$-locally separable for all Lévy measures $\nu$ on $E$.

(3) We denote by

$$K_0^* := K_0^*(E) := \{ K \in K_0 : E_K \text{ separable with respect to } \| \cdot \|_K \}.$$  

(4) If for every $K \in K_0(E)$ there exists $K' \supseteq K$ with $K' \in K_0^*$, we call $E$ separably extendable.

**Lemma 7.3.5.** If $E$ is separably extendable, it is locally separable.

*Proof.* For every Lévy measure $\nu$ there is a $K \in K_0$ with $\nu(K^c) < \infty$. By separable extendability of $E$ there is a $K' \supseteq K$ with $K' \in K_0^*$. \hfill \Box

In Appendix B we give sufficient conditions and examples for separably extendable spaces.

**Lemma 7.3.6.** Let $\tau$ denote the given topology on $E$ and let $\tau'$ be another comparable locally convex Suslin topology on $E$. A measure $\rho$ is a Gaussian measure with respect to $(E, \tau)$ if and only if it is a Gaussian measure with respect to $\tau'$.

*Proof.* Let $\rho$ be Gaussian on $\mathcal{B}(E_\tau) = \mathcal{E}(E,E'_\tau)$. By [Bog91, Proposition 2.2.10], this is equivalent to the statement that for the map $\psi : X \times X \to X$, defined by $(x, y) \mapsto x \sin \varphi + y \cos \varphi$ it holds that $(\rho \otimes \rho)(\psi^{-1}(B)) = \rho(B)$ for all $B \in \mathcal{B}(E_\tau) = \mathcal{B}(E_{\tau'}) = \mathcal{E}(E,E'_{\tau'})$. This means, $\rho$ is Gaussian on $E_{\tau'}$. \hfill \Box

**Lemma 7.3.7.** Let $\tau$ denote the given topology on $E$ and let $\tau'$ be another comparable locally convex Suslin topology on $E$. $\tau$ and $\tau'$ need not be complete. A measure $\mu \in M^1(E)$ is a Poisson measure with respect to $\tau$ if and only if it is a Poisson measure with respect to $\tau'$.

*Proof.* As recalled above, the Borel-$\sigma$-algebras of both topological spaces coincide and one obtains the result by considering the setwise converging series (6.2.1) which only depends on the Borel structure. \hfill \Box

**Lemma 7.3.8.** Let $\tau$ denote the given topology on $E$ and let $\tau'$ be another comparable locally convex Suslin topology on $E$. A measure $\nu$ is a Lévy measure with respect to $(E, \tau)$ if and only if it is a Lévy measure with respect to $\tau'$. 

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Proof. The Borel-σ-algebras of both topological spaces coincide. Nevertheless, the definition of Lévy measures uses the topology for characterization, so there is actually something to show. Without loss of generality assume $\tau' \supseteq \tau$. Then, if $\nu$ is a Lévy measure on $(E, \tau')$, it follows from shift tightness of the defining family of the Poisson measures in $(E, \tau')$ and thus $(E, \tau)$ that it is a Lévy measure on $(E, \tau)$.

Conversely, if $\nu \in \mathcal{M}(E)$ is a Lévy measure with respect to $\tau$, then there exists an infinitely divisible measure $\mu := \mathcal{D}(\nu) \in \mathcal{M}_1(E)$. Now we consider $\tau'$ as underlying topology. Note that infinite divisibility depends only on the Borel structure. Because $\mu$ is infinitely divisible, the Lévy-Khintchine decomposition (w.r.t. $\tau'$) (Theorem 6.2.5) provides a unique Lévy measure $\nu'$. It follows from the converse implication that $\nu' = \nu$.

7.3.2 Restriction to a subspace.

We begin with some lemmas, introduce the notion of local reducibility and present the main results of this section, Theorem 7.3.14 and Proposition 7.3.15.

Lemma 7.3.9. Let $M \subseteq \mathcal{M}_1(E)$ be a family of measures with the following properties:

1. There is a measurable linear subspace $L \subseteq E$ such that $\mu(L) = 1$ for all $\mu \in M$.
2. $M$ is shift tight, i.e. for $\varepsilon > 0$ there is a compact set $K \subseteq E$ such that for all $\mu \in M$ there exist $x_\mu \in E$ with

$$\mu \ast \delta_{x_\mu}(K) > 1 - \varepsilon.$$ 

Then, there exist $x_\mu^0 \in L$ such that

$$\mu \ast \delta_{x_\mu^0}(2K^\circ\circ) > \mu \ast \delta_{x_\mu^0}(2(K^\circ\circ) \cap L) > 1 - \varepsilon,$$

where $K^\circ\circ$ denotes the bipolar of $K$.

Proof. The bipolar is the closed absolutely convex hull of $K$. Without loss of generality we can therefore assume that $K$ is absolutely convex as the absolutely convex hull of a compact set in a complete locally convex space is compact, cf. [Jar81, Proposition 6.7.2]. We have that $\mu(L) = 1$ for every $\mu \in M$ and therefore, $\mu \ast \delta_{x_\mu}(L - x_\mu) = 1$. We define

$$K_\mu := (L - x_\mu) \cap K, \quad \text{and} \quad K_{\mu}^\circ := -K_\mu = (L + x_\mu) \cap K,$$

and obtain $\mu \ast \delta_{x_\mu}(K_\mu) = \mu \ast \delta_{x_\mu}(K) > 1 - \varepsilon$. We show that there exists a $\pi \in K_{\mu}^\circ$ with $x_\mu - \pi \in L$. This is the case precisely if $(x_\mu + K_{\mu}) \cap L \neq \emptyset$. One obtains the claim.
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by $\mu \ast \delta_{x_\mu}(K_\mu) = \mu(x_\mu + K_\mu) = \mu((x_\mu + K_\mu) \cap L) > 1 - \varepsilon$. So, for every $\mu$, one can find such an element, we call it $\pi_\mu \in K_\mu$ such that $x^0_\mu := x_\mu - \pi_\mu \in L$. The following properties hold for every $\mu \in M$:

1. $\mu \ast \delta_{x_\mu - \pi_\mu}(K_\mu + \pi_\mu) > 1 - \varepsilon$.
2. $K_\mu + \pi_\mu \subset 2K = K + K$.
3. $K_\mu + \pi_\mu \subset L$, because $\pi_\mu + K_\mu \subset \pi_\mu - x_\mu + L = L$ by definition of $K_\mu$.

In particular, $(\mu \ast \delta_{x_\mu})(2K \cap L) \geq \mu \ast \delta_{x_\mu}(K_\mu + \pi_\mu) > 1 - \varepsilon$ for all $\mu \in M$.

**Lemma 7.3.10.** Let $\mu \in \mathcal{I}(E)$ and $L$ a linear subspace of $E$ with $\mu(L) = 1$. Then, $\mu_{1/n}(L) = 1$ for every $n \in \mathbb{N}$. In particular, $\mu\|_L$ is infinitely divisible on $L$.

*Proof.* Set $\varrho = \mu_{1/n}$ and assume a set $C \in \mathcal{B}(E)$ with $C \cap L = \emptyset$ and $\varrho(C) > 0$. Then, $\varrho * \varrho(C + C) \geq \varrho(C)^2 > 0$. If $(C + C) \cap L = \emptyset$, then $\varrho^2(C + C) > 0$ and one can carry on with induction. If there is a $k \in \{1, \ldots, n-1\}$ such that $\varrho^k$ has positive measure on $C_k := C_{k-1} + C$ (with $C_1 := C$) and $C_k \cap L \neq \emptyset$, there are two cases: Either $\varrho^k(C_k \cap L) = 0$. Then, we define $C'_k := C_k \setminus L$ and $C_{k+1} := C'_k + C$, consider $\varrho^{k+1}(C_{k+1})$ and carry on as above. We obtain a contradiction for $\mu(C_n) = \varrho^{*n}(C_n) > 0$. However, if the intersection has positive measure $\varrho^k$, we obtain that $\varrho^k(L) > 0$ and consequently $\varrho * \varrho^k(C + L) \geq \varrho(C) \varrho^k(L) > 0$.

Now, let $n = kl + m$ for $m < k$ and some integer $l$. By construction, $C_m \cap L \neq \emptyset$ and $\varrho^m(C_m) > 0$ hold, in particular, $(C_m + L) \cap L = \emptyset$. We obtain $\mu(C_m + L) = \varrho^m * (\varrho^k)^l(C_m + L) \geq \varrho^m(C_m)(\varrho^k(L))^l > 0$, a contradiction.

**Definition 7.3.11.** A Lévy measure $\nu \in \mathcal{M}(E)$ is called **locally reducible** if there exists a compact set $K \in \mathcal{K}_0^\ast$ with $\nu(K^c) < \infty$ such that there exists a generalized Poisson exponential of $\nu\|_K$ with $\hat{\nu}\|_K(E_K) = 1$. In this case, $K$ is called $\nu$-reducing. An infinitely divisible distribution is **locally reducible** if its corresponding Lévy measure has this property.

The following results are immediate:

**Lemma 7.3.12.**

1. If $\nu$ is locally reducible on $E$, then $E$ is $\nu$-locally separable.

2. If $\nu$ is locally reducible and $K \in \mathcal{K}_0^\ast$ is $\nu$-reducing, then every $H \in \mathcal{K}_0^\ast$ with $H \supseteq K$ is $\nu$-reducing.

The following example is inspired by [Sch73, Theorem 7, p 112].
Example 7.3.13. If \( E \) is a separable Fréchet space, all \( \mu \in \mathcal{I}(E) \) are locally reducible. If \( E \) is an inductive limit of separable Fréchet spaces \( E_n, n \in \mathbb{N} \), such that \( E_n \subseteq E_{n+1} \) and every compact set of \( E \) is a compact set in some \( E_n \), then all \( \mu \in \mathcal{I}(E) \) are locally reducible. An example is the space of test functions \( \mathcal{D}(\Omega), \Omega \subseteq \mathbb{R}^n \) open.

Proof. The first assertion follows from [Bog91, Theorem 3.6.5], stating that in every Fréchet space a Radon measure is concentrated on a separable Banach subspace whose closed unit ball is compact in \( E \).

For the second assertion, note that \( E \) is a complete locally convex Suslin space (Suslin spaces are closed under strict inductive limits, [Sch73, p. 111]). We find a set \( K_0 \in \mathcal{K}_0(E) \) such that \( \nu(K_0^c) < \infty \). By assumption, it is in some Fréchet space \( F = E_n \).

In \( F \), one finds a larger compact disk \( K_1 \in \mathcal{K}_s(F) \), cf. Corollary 7.4.7. It is not hard to see \( \mathcal{E}(\nu|_{K_1})(E_{K_1}) = 1 \). It suffices to construct continuous functions \( f \) with \( f = 0 \) on the closure and one on a compact set of suspected positive measure and find a contradiction to weak convergence, cf. to the proof of Theorem 7.3.14, (6) \( \Rightarrow \) (2) below. The closure is entirely contained in \( F \) as \( F \) is closed in \( E \). Therefore, \( \mathcal{E}(\nu|_{K_1}) \) is a finite measure on \( \mathcal{B}(F) \) (which is nothing else than the trace \( \sigma \)-algebra of \( E \)) for which there exists a compact disk \( C \in \mathcal{K}_s(F) \subseteq \mathcal{K}_s(E) \) such that \( \mathcal{E}(\nu|_{K_1})(E_C) = 1 \), which is the assertion. The last part is [Jar81, Example 4.6.3].

Theorem 7.3.14 (Equivalences for local reducibility). Let \( \mu \in \mathcal{I}(E) \) and \( \nu \in \mathcal{M}(E) \) be its corresponding Lévy measure. Then the following assertions are equivalent.

1. \( \mu \) is locally reducible.
2. \( \mathcal{E}(\nu|_K)(E_K) = 1 \) for some \( K \in \mathcal{K}_0^s(E) \) with \( \nu(K^c) < \infty \), i.e. \( \nu \) is locally reducible.
3. The function
   \[
   \varphi(a) = \exp \left( \int_{E_K} e^{i(x,a)} - 1 - i(x,a) \mathbb{1}_K(x) \, d\nu(x) \right), \quad a \in E',
   \]
   is a Fourier transform of some probability measure on \( \mathcal{B}(E_K) \) for some \( K \in \mathcal{K}_0^s(E) \) with \( \nu(K^c) < \infty \).
4. The restriction of \( \nu|_K \) to the subspace \( (E_K, \| \cdot \|_K) \) is a Lévy measure for some \( K \in \mathcal{K}_0^s(E) \) with \( \nu(K^c) < \infty \).
5. The restriction of \( \nu|_K \) to the subspace \( (E_K, \tau) \) is a Lévy measure for some \( K \in \mathcal{K}_0^s(E) \) with \( \nu(K^c) < \infty \).
6. For some \( K \in \mathcal{K}_0^s(E) \) with \( \nu(K^c) < \infty \) the following holds: There are \( x_n \in E \) such that for every \( \varepsilon > 0 \) there exists an \( n \in \mathbb{N} \) such that for all \( \mu \leq \nu|_K \) one has
   \[
   e(\mu) * \delta_{x_n}(n \cdot K) > 1 - \varepsilon.
   \]
Furthermore, if there is $K \in \mathcal{K}_0^+(E)$ such that one of the assertions (2)–(6) holds, one can take the same set also in all other assertions.

**Proof.** (1) $\iff$ (2) holds by definition, and (4) $\iff$ (5) follows by taking a suitable compact set $K$ by Lemma 7.3.8. 

(3) $\Rightarrow$ (4): In a Banach space, $\nu\|_{E_K}$ is a Lévy measure if and only if $\varphi$ is the characteristic functional of a some probability measure, cf. [Hey10, Theorem 3.4.9], where $\varphi$ is evaluated in all $a \in E_K'$. The set $E'_n\|_{E_K}$ separates the points of $E_K$ and therefore, $\varphi$ uniquely determines a measure on $\mathcal{B}(E_K) = E(E_K, E'_n\|_{E_K})$. This implies that $\nu\|_{E_K}$ and hence $(\nu|_K)\|_{E_K}$ is a Lévy measure on $E_K$. This yields assertion (4). 

(4) $\Rightarrow$ (3): Again using [Hey10, Theorem 3.4.9], it suffices to note $E'_\|_{E_K} \subseteq E_K'$ and $\nu(K^c) < \infty$. 

(2) $\Rightarrow$ (3): By assumption, $\widetilde{e}(\nu|_K)(E \setminus E_K) = 0$ and $\widetilde{e}(\nu|_K)$ and $\widetilde{e}(\nu|_K)\|_{E_K}$ have the same Fourier transform if we use the set of continuous functionals $E'$ generating $\mathcal{B}(E)$ and $\mathcal{B}(E_K)$. One obtains

$$
\widetilde{e}(\nu|_K)\|_{E_K}(a) = \exp \left( \int_{E_K} e^{i(x,a)} - 1 - i(x,a)1_{E_K} \, d\nu|_K(x) \right), \quad a \in E'_\|_{E_K}
$$

which yields (3) if one takes the convolution of this measure with the Poisson measure of $(\nu|_{E_K \setminus E})\|_{E_K} \in \mathcal{M}(E_K)$.

(4) $\Rightarrow$ (6): If $\nu' := (\nu|_K)\|_{E_K}$ is a Lévy measure on $E_K$, there exist $x_\mu \in E_K \subseteq E$ such that the family $(e(\mu) * \delta_{x_\mu})_{\mu \leq \nu'}$ is uniformly tight, i.e., for every $\varepsilon$ there exists a $\parallel \cdot \parallel_K$-norm compact set $H_\varepsilon$ with $(e(\mu) * \delta_{x_\mu})(H'_\varepsilon) > 1 - \varepsilon$ for all $\mu \leq \nu'$. As every compact set is bounded, there exists an $n(\varepsilon) \in \mathbb{N}$ with $H_\varepsilon \subseteq n(\varepsilon) \cdot K$ which implies assertion (6).

(6) $\Rightarrow$ (2): If $\nu$ is finite, we have $e(\nu)(E \setminus E_K) = 0$ because $\nu^n(E \setminus n \cdot K) = 0$ due to $n \cdot K \subseteq E_K$. So let $\nu$ be a Lévy measure which is not finite. The measure $\nu|_K$ is a Lévy measure on $E$. By assumption, there exist shifts $x_\mu \in E$ such that the family of shifted Poisson measures satisfies the $n \cdot K$-tightness condition in (6). By Lemma 7.3.9, the shifts $x_\mu$ can be taken in $E_K$ without loss of generality. As $\tilde{e}(\nu|_K)$ is an accumulation point of the family $(e(\mu) * \delta_{x_\mu})_{\mu \leq \nu|_K}$, for all $\varepsilon > 0$, all $f \in C^b(E)$ and all $\mu_0$ there exists a measure $\mu \geq \mu_0$ such that

$$
|\tilde{e}(\nu|_K)(f) - e(\mu) * \delta_{x_\mu}(f)| < \varepsilon.
$$

(7.3.1)

Assuming that there exists a Borel set $B \subseteq E \setminus E_K$ with positive measure $\tilde{e}(\nu|_K)$, we also find a compact set $C \subseteq B$ with positive measure. There exists a continuous function $g: E \to [0,1]$ such that $g = 0$ on the closed set $n \cdot K$ and $g = 1$ on the compact set $C$ due to complete regularity of $E$. We have $\tilde{e}(\nu|_K)(g) \geq \tilde{e}(\nu|_K)(C) > 0$. Furthermore, we see that $e(\mu) * \delta_{x_\mu}(E_K \setminus n \cdot K) \leq \varepsilon$ for $n$ large enough which can be chosen independently of $\mu$ and $x_\mu$ by assumption. But for all $\varepsilon > 0$ one finds a $\mu$ and
for \( g \) constructed as above. This is a contradiction to the claim that the generalized Poisson measure had positive measure on \( E \setminus E_K \).

\[ \text{Proposition 7.3.15 (Reducing operators and identification of measures).} \]

Let \( K \in \mathcal{K}_0(E) \).

(1) The operators

\[ \cdot^0 : \mathcal{M}(E_K) \rightarrow \mathcal{M}(E) \quad \text{and} \quad \cdot \|_{E_K} : \mathcal{M}_K(E) \rightarrow \mathcal{M}(E_K) \]

are bijective and inverse to each other.

(2) The operators

\[ \cdot^0 : \mathcal{M}^0(E_K) \rightarrow \mathcal{M}^0(E) \quad \text{and} \quad \cdot \|_{E_K} : \mathcal{M}^0_K(E) \rightarrow \mathcal{M}^0(E_K) \]

are bijective and inverse to each other.

(3) A measure \( \mu \in \mathcal{M}^1_K(E) \) is a Poisson distribution on \( E \) if and only if \( \mu \|_{E_K} \in \mathcal{M}^1(E_K) \) is also a Poisson distribution on \( E_K \).

(4) If a measure \( \nu \in \mathcal{M}_K(E) \) is a reducible Lévy measure and \( K \) is \( \nu \)-reducing, then \( \nu \|_{E_K} \in \mathcal{M}(E_K) \) is a Lévy measure on \( E_K \). Conversely, if \( \nu' \in \mathcal{M}(E_K) \) is a Lévy measure on \( E_K \), \( (\nu')^0 \in \mathcal{M}(E) \) is a Lévy measure on \( E \).

(5) Let \( \varrho \in \mathcal{M}^1(E) \) and \( \varrho(K) > 0 \). Then, \( \varrho \) is a Gaussian measure on \( E \) if and only if \( \varrho \|_{E_K} \) is Gaussian measure on \( E_K \).

(6) \( \mu \in \mathcal{I}(E) \cap \mathcal{M}^1_K(E) \) if and only if \( \mu \in \mathcal{I}(E_K) \).

\[ \text{Proof.} \quad (1) \text{ is clear by Lemma 7.3.3 (2).} \]

(2) is a consequence of Lemma 7.3.3 (2) and the fact that finite measures on comparable Suslin topologies coincide, cf. [Sch73, Corollary 2, p. 124].

(3) follows from (1) and (2) and Lemma 7.3.7 and the following consideration: If
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\( \mu(E_K) = 1 \) and \( \mu = e(\nu) \) for \( \nu \in \mathcal{M}^\delta(E) \), then one obtains that \( \nu(E \setminus E_K) = 0 \) using the series representation. In fact, one has that \( e(\nu)\|E_K = e(\nu)\|E_K) \).

(4) follow from (1) and (2), Theorem 7.3.14 and Lemma 7.3.8.

(5) uses the zero-one law for Gaussian measures, stating that if a compact set \( K \in \mathcal{K}_0 \) has positive measure, then its linear hull must have full measure, cf. [Bog91, Theorem 2.5.5]. Thus, we can consider the restriction of \( \varrho \) to \( E_K \) and use Lemma 7.3.6 exploiting the comparable Suslin topologies.

(6) follows from (1) and (2) and Lemma 7.3.10.

If a Lévy measure is locally reducible, we obtain a simpler characterization.

**Proposition 7.3.16.** A measure \( \nu \in \mathcal{M}(E) \) is a locally reducible Lévy measure with \( \nu \)-reducing set \( K \in \mathcal{K}_0^\delta(E) \) if and only if

(i) \( \nu(\{0\}) = 0 \),

(ii) \( \nu\vert_{(\alpha K)'} < \infty \) for some (all) \( \alpha > 0 \), and

(iii) the family \( \{e(\nu\vert_{K^{\delta_nK}})\|E_K : n \in \mathbb{N}\} \) is shift tight in \( (E_K, \| \cdot \|_K) \) for every (some) positive null sequence \( (\delta_n)_n \).

**Proof.** \( \implies \): If \( \nu \) is locally reducible with \( \nu \)-reducing set \( K \in \mathcal{K}_0^\delta \) then \( (\nu\vert_K)\|_{E_K} \) is a Lévy measure on the separable Banach space \( E_K \) by Proposition 7.3.15. By [Hey10, Proposition 3.4.9] and Prokhorov’s theorem, we have that \( (\nu\vert_K)\|_{E_K} \) satisfies (i) \( (\nu\vert_K)\|_{E_K}(\{0\}) = 0 \), (ii) \( (\nu\vert_K)\|_{E_K}((\delta K)') < \infty \) and (iii) for all (some) null sequences \( \delta :=(\delta_n)_n \) the set \( M(\delta, K) \) is shift tight w.r.t. \( E_K \). The assertion follows from taking null extensions to the original space and \( \nu(K') < \infty \).

\( \iff \): Let \( \nu \) satisfy (i)-(iii), then \( (\nu\vert_K)\|_{E_K} \) is a Lévy measure on \( E_K \) by [Hey10, Proposition 3.4.9]. In particular, \( M(\delta, K) := \{e(\nu\vert_{K^{\delta_nK}})\|E_K : n \in \mathbb{N}\} \) is shift tight in \( E_K \) and, a fortiori w.r.t. \( (E_K, \tau) \) thus \( E \). We note that \( M' := M(\delta, K)^0 \ast e(\nu\vert_{K'}) \) (elementwise convolution) is shift tight in \( E \): If for \( \epsilon \in (0, 1) \) the set \( K_{\epsilon/2} \in \mathcal{K}_0(E) \subseteq \mathcal{K}_0(E) \) satisfies \( \mu_0(E \setminus K_{\epsilon/2}) = \mu(E_K \setminus K_{\epsilon/2}) < \epsilon/2 \) for all \( \mu \in \mathcal{M} \), then \( H_{\epsilon/2} \in \mathcal{K}_0(E) \) satisfies \( e(\nu\vert_{K'})(H_{\epsilon/2}) < \epsilon/2 \), then

\[
\mu \ast e(\nu\vert_{K'})((K_{\epsilon/2} + H_{\epsilon/2})^c) = 1 - \mu \ast e(\nu\vert_{K'})((K_{\epsilon/2} + H_{\epsilon/2})^c)
\leq 1 - (K_{\epsilon/2}) e(\nu\vert_{K'})(H_{\epsilon/2})
< 1 - (1 - \epsilon/2)^2 < \epsilon.
\]

Furthermore, \( \nu = \sup_{\mu \in M'} \mu \in \mathcal{M}(E) \) which proves that \( \nu \) is a Lévy measure on \( E \).

The first remark gives local reducibility of \( \nu \) to \( E_K \).

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7.3 Separable Banach subspaces

7.3.3 Zero-one laws and reducibility

Using a zero-one law of A. Janssen (cf. Theorem 7.3.17) we will see that on separably extendable spaces all infinitely divisible measures are locally reducible.

**Theorem 7.3.17** (Janssen, [Jan82, Theorem 9]). Let \( \nu \in \mathcal{M}(E) \) be a Lévy measure and \( \mu := \tilde{c}(\nu) \) its generalized Poisson exponential. If \( H \) is a linear subspace of \( E \) with \( \nu(H^c) = 0 \) and \( x \in E \), then \( \mu(x + H) \in \{0, 1\} \).

**Lemma 7.3.18.** If \( K_1, K_2 \in \mathcal{K}_0^0(E) \) then \( K_1 + K_2 \in \mathcal{K}_0^0(E) \).

**Proof.** It is easy to check that \( K_1 + K_2 \in \mathcal{K}_0 \), so that it remains to show the separability of \( E_{K_1+K_2} \). By assumption, \( E_{K_i} \) has a countable dense subset \( E_i^0 \), \( i = 1, 2 \). We show that \( E^0 := E_1^0 + E_2^0 \) is dense in \( E_{K_1+K_2} \). So take \( x \in E_{K_1+K_2} \). There exist \( x_i \in E_{K_i}, i = 1, 2 \) such that \( x = x_1 + x_2 \). Furthermore, there are \( x_i^0 \in E_i^0 \) with \( \|x_i - x_i^0\|_{K_i} \leq \varepsilon \), \( i = 1, 2 \). Then,

\[
\|x - x_0\|_{K_1+K_2} \leq \|x_1 - x_1^0\|_{K_1+K_2} + \|x_2 - x_2^0\|_{K_1+K_2} \\
\leq \|x_1 - x_1^0\|_{K_1} + \|x_2 - x_2^0\|_{K_2} < \varepsilon.
\]

**Theorem 7.3.19.** Let \( E \) be separably extendable. Then, the following holds:

1. For every generalized Poisson exponential \( \mu \) with Lévy measure \( \nu \) satisfying \( \nu(K_0^0) = 0 \), \( K_0 \in \mathcal{K}_0(E) \), there exists \( K \in \mathcal{K}_0^0(E) \) such that \( \mu(E_K) = 1 \).

2. Every infinitely divisible measure \( \mu \in \mathcal{I}(E) \) is locally reducible.

**Proof.** (1) Let \( \mu = \tilde{c}(\nu) \). As \( E \) is separably extendable and \( \mu \) is tight, there exists \( K_1 \in \mathcal{K}_0^0(E) \) with \( \mu(E_{K_1}) \geq \mu(K_1) > 0 \). Furthermore, there exists a set \( K_2 \in \mathcal{K}_0^0(E) \) with \( K_0 \subseteq K_2 \) and \( \nu(K_2^c) = 0 \). We set \( K := K_1 + K_2 \in \mathcal{K}_0^0(E) \) by Lemma 7.3.18. Then, we have that \( \nu(E_K) \leq \nu(K_2) = 0 \) and \( \mu(E_K) \geq \mu(K_1) > 0 \). Theorem 7.3.17 implies that \( \mu(E_K) = 1 \). In other words, \( \mu \) is locally reducible.

(2) Let \( \mu \) be infinitely divisible with Lévy measure \( \nu \). According to (1), there exists \( K_0 \in \mathcal{K}_0(E) \) such that \( \nu(K_0) \) is a finite measure, and a set \( K \supseteq K_0 \) with \( K \in \mathcal{K}_0^0(E) \) such that \( \tilde{c}(\nu|_{K_0})(E_K) = 1 \). Noting that

\[
\tilde{c}(\nu|_{K})(E_K) = e(\nu|_{K\setminus K_0}) \cdot \tilde{c}(\nu|_{K_0})(E_K + E_K) \geq e(\nu|_{K\setminus K_0})(E_K) \tilde{c}(\nu|_{K_0})(E_K) = 1
\]

we obtain the assertion.

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In this section we give sufficient conditions for spaces to be separably extendable and conditions for local reducibility.

**Proposition 7.4.1.** Fréchet spaces are separably extendable.

**Proof.** The proof is essentially given in [Bog91, Theorem 3.6.5] but presented here for convenience: If \( K \in K_0 \) there exists \( A \in K_0 \) such that \( K \) is compact in \( E_A \), cf. [HNM81, Lemma p.18]. Again, in \( E_A \) there exists \( C \in K_0(E_A) \subseteq K_0(E) \) with \( K \) compact in \( E_C \) and \( C \) is compact in \( E_A \). A factorization lemma of Davis-Fiegel-Johnson-Pełchiński [DFJP74, Corollary 1] for weakly compact operators provides a Banach space \( Y \) which is reflexive and continuously embedded into \( E_A \), and \( C \) is bounded in \( Y \). The continuity of \( E_C \to Y \) yields that \( K \) is compact in \( Y \). Taking the closure \( L \) in \( Y \) of the linear hull of \( K \) yields a reflexive subspace of \( Y \) which is separable as it is the closure of the image of the compact mapping \( J: E_K \to Y \), where \( J \) is the natural embedding). By blowing up by a suitable factor, the closed unit ball \( K_0 \) of \( L \) can be chosen such that \( K \) is contained in \( K_0 \) by continuity. Finally, \( K_0 \) is compact in \( E \) as reflexivity implies that \( K_0 \) is weakly compact in \( L \) and therefore weakly compact in \( E \) by continuity of the natural embedding. But this implies that \( K_0 \) is weakly closed and by convexity and precompactness in \( E \) we have \( K_0 \in K_s(E) \). \( \qed \)

The following example establishes the connection of our approach to the work of Üstünel, cf. [Üst84], who considered Lévy processes with values in strong duals of nuclear spaces which are nuclear and Suslin. Under the condition that \( E \) is barreled, these spaces are separably extendable:

**Example 7.4.2.** Let \( E' \) be a nuclear Suslin space and a strong dual of a separable nuclear barreled space \( E \). Then, if \( K \) is a compact set in \( E' \), there exists an absolutely convex compact set \( S \supset K \) such that \( E'_S \) is a separable Hilbert space. Thus, \( E' \) is separably extendable.

**Proof.** In a nuclear space there exists a neighbourhood base \( U \) such that for all \( U \in U \) the completion of the space \( E/p^{-1}(\{0\}) \) is a separable Hilbert space \( E(U) \). Its dual space can be identified with \( E_u^* \), where \( U^* \) is the polar of \( U \). Define \( K' := \{U^*: U \in U\} \) which is a fundamental system of closed bounded sets in \( E' \) because \( E \) is barreled, cf. [Sch71, 5.2, p. 141]. As \( E' \) is nuclear and therefore, all bounded sets are precompact, cf. [Sch71, p. 101, Corollary 2], \( K' \) consists of compact sets only. Choosing a compact set \( K \) in \( E' \) one finds an \( S \in K' \) with \( S = U^*, U \in U \), such that \( K \subseteq S \). Consequently, \( K \subseteq E_S \cong (E(U))' \) which is a separable Hilbert space. \( \qed \)

**Remark 7.4.3.** In the proof of Theorem III.1 of [Üst84], it is claimed that for the given set \( K \in K_0 \) one can choose \( S \in K' \) such that \( K \subseteq S \) but no justification is
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given. The missing assumption of barreledness of the predual seems essential in order to obtain that \( K' \) is a fundamental system of bounded sets – what is in fact used at this point.

### 7.4.1 A SUFFICIENT CONDITION FOR SEPARABLE EXTENDABILITY

The construction in Proposition 7.4.1 is only known for Fréchet spaces. If for every compact disk \( K \) in \( E \) one can find a larger compact disk \( B \) with compact embeddings \( E_K \hookrightarrow E_B \hookrightarrow E \) one obtains a similar result.

**Proposition 7.4.4.** Let \( K \in K_0(E) \). If there exists a compact disk \( B \subseteq E \) containing \( K \) and such that the canonical injection \( J: E_K \to E_B \) is compact, then there is also a compact disk \( K_0 \in K_0(E) \) containing \( K \). In particular, if \( \nu(K^c) < \infty \), \( E \) is \( \nu \)-locally separable.

**Proof.** First we note that \( K \) is compact in \( E_B \) as it is precompact by definition and closed by virtue of continuity of \( E_B \), \( \hookrightarrow E \). The Banach space \( E_B \) allows to find the desired compact set \( K_0 \in K_0(E_B) \subseteq K_0(E) \) by Proposition 7.4.1.

**Definition 7.4.5 (\( \mathcal{G} \)-co-Schwartz and \( \mathcal{G} \)-conuclear spaces).**

1. A system \( \mathcal{G} \) of subsets of \( E \) is **directed upwards by inclusion** if for \( A, B \in \mathcal{G} \) there exists \( C \in \mathcal{G} \) with \( A \cup B \subseteq C \).

2. A collection of closed disks \( \mathcal{G} \) is a bornological basis if it is directed upwards by inclusion and if \( \alpha A \in \mathcal{G} \) for all \( \alpha \in \mathbb{R} \) and \( A \in \mathcal{G} \).

3. Let \( \mathcal{G} \) be a bornological basis. A locally convex Hausdorff space \( E \) is called \( \mathcal{G} \)-co-Schwartz resp. \( \mathcal{G} \)-co-nuclear if for every \( B \in \mathcal{G} \) there exists \( C \in \mathcal{G} \) such that the canonical embedding

\[
J_{BC}: \tilde{E}_B \to \tilde{E}_C
\]

is compact resp. nuclear where \( \tilde{E}_B \) denotes the completion of the normed space \( E_B \) if necessary.

Any \( \mathcal{G} \)-conuclear space is \( \mathcal{G} \)-co-Schwartz. If \( \mathcal{G} = \mathcal{B} \), the system of closed disks in \( E \), \( \mathcal{B} \)-co-Schwartz resp. \( \mathcal{B} \)-conuclear is frequently called **co-Schwartz** resp. conuclear. These notions are well-established, cf. [HNM81, Jar81].

It is easy to verify that the set of compact disks \( K_0 \) forms a bornological basis in \( E \). The theorem of Minlos for cylindrical measures holds for \( K_0 \)-conuclear spaces, cf. [Sch73, Theorem 2, p. 233]. In the following, we will mostly use the notion of a \( K_0 \)-co-Schwartz space, i.e. \( \mathcal{G} = K_0 \). In this case, \( E_B \) and \( E_C \) from the previous definition are Banach spaces.

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Corollary 7.4.6. \( K_0 \)-co-Schwartz spaces are separably extendable.

Proof. By Proposition 7.4.4, for every compact disk \( K \) one can find a larger compact disk \( K_0 \in K_0^0(E) \).

The set \( K_0 \supseteq K \) in the proof of Proposition 7.4.1 was constructed in a way that \( K \) is even compact in \( E_{K_0} \). Therefore we obtain

Corollary 7.4.7. Fréchet spaces are \( K_0 \)-co-Schwartz.

Lemma 7.4.8. Co-Schwartz spaces are \( K_0 \)-co-Schwartz.

Proof. If \( E \) is a co-Schwartz space, it is quasi-complete, cf. [HNM81, Chapter 1, Theorem (4d)]. Let \( K \in K_0(E) \). It suffices to show that there exists \( B \in K_0(E) \) such that the canonical embedding \( J_{KB} \) is compact. For \( K \) one finds a larger (not necessarily compact) disk \( C \) such that the extension of the canonical embedding \( J_{KC} \) is compact.

Without loss of generality \( C \) is closed (e.g. take the closure of a suitable disk), so we assume that \( C \) be closed, thus complete by quasi-completeness of \( E \). Its linear hull \( E_C \) is a Banach space. In particular, it is \( K_0 \)-co-Schwartz by Corollary 7.4.7. Therefore, one finds a compact disk \( B \supseteq K \) in \( E_C \) (and therefore in \( E \) by the continuous injection) and the assertion follows.

Remark 7.4.9. Interestingly, although we need our assumptions for different purposes, Dettweiler posed essentially the same two conditions in [Det76b, Section 3]: In \( E \) there should exist a fundamental system \( K_{H}^* \) of \( K_H \) of compact Hilbert disks (\( E_K \) is a separable Hilbert space for all \( K \in K_H^* \)) – this is our notion of separable extendability for Hilbert disks. A second condition requests that for every \( K \in K_H^* \) there is an \( L \in K_H^*, K \subseteq L \), such that \( \iota: E_K \rightarrow E_L \) is compact, i.e. \( E \) is a \( K_H \)-co-Schwartz space which is a stricter notion of a \( K_0 \)-co-Schwartz space.

Theorem 7.4.10 (Comparable Topologies and Reduction Properties). For a complete locally convex Suslin space \( (E, \tau) \), let a locally convex Suslin topology \( \tau' \) satisfy either

(i) \( \tau' \) is stronger than \( \tau \) or

(ii) \( \tau' \) is weaker than \( \tau \) and compatible with duality.

Consider the following assertions.
7.4 Some functional analysis

(a) If \((E, \tau')\) is locally separable, then \((E, \tau)\) is locally separable.

(b) If \((E, \tau')\) is \(K_0\)-co-Schwartz, then \((E, \tau)\) is separably extendable.

(c) If a Lévy measure on \((E, \tau')\) is locally reducible, it is locally reducible on \((E, \tau)\).

Then: (i) implies (a) and (c); and (ii) implies (b) and (c).

**Proof.** (i) implies (a): Lévy measures on comparable Suslin topologies coincide (cf. Lemma 7.3.8), so that we consider \(\nu\) on the finer topology \(\tau'\). By local separability of \((E, \tau')\) one finds \(K \in K_0^s(\tau') \subseteq K_0^s(\tau)\) with \(\nu(K^c) < \infty\) which is the assertion.

(i) implies (c): If \(\nu\) is a Lévy measure, it is a Lévy measure on \((E, \tau')\), where \(\tau' \supseteq \tau\) is a locally convex Suslin topology, cf. Lemma 7.3.8. There exists \(K \in K_0^s(E, \tau')\) with \(e(\nu|_K)(E_K) = 1\) and as \(K \in K_0^s(E, \tau)\), the assertion follows.

(ii) implies (b): Let \(K \in K_0(E, \tau) \subseteq K_0(E, \tau')\). As \((E, \tau')\) is \(K_0\)-co-Schwartz, one finds a set \(K' \in K_0(E, \tau')\) such that \(K \subseteq K'\) and \(K\) is compact in \(E_{K'}\). The set \(K'\) is closed and bounded in \((E, \tau)\) by compatibility of the topologies, therefore \(E_{K'}\) is a Banach space. By Proposition 7.4.1 there exists \(B \in K_0^s(E_{K'}) \subseteq K_0^s(E)\) with \(K \subseteq B\) which is the assertion.

(ii) implies (c): By local reducibility of \((E, \tau')\) for a Lévy measure \(\nu\), one finds \(K \in K_0^s(E, \tau')\) and \(\nu|_K\|_{E_K}\) is a Lévy measure by Theorem 7.3.14 and \(e(\nu|_K)(E \setminus E_K) = 0\). By compatibility of the topologies, \(K\) is a closed disk in \((E, \tau)\) and its linear hull is a separable Banach space with respect to the unit ball \(K\). By [Bog91, Theorem 3.6.5], for the Radon measure \(e(\nu|_K)|_{E_K}\) one can find a compact disk \(B \in K_0^s(E_K) \subseteq K_0^s(E, \tau)\) such that \(E_B\) has full measure. As \(B\) is compact in \((E, \tau)\), the Banach space \(E_B\) is the desired Banach support of the generalized Poisson exponential.

**Corollary 7.4.11.** A Lévy measure is locally reducible on a complete locally convex Suslin space \((E, \tau)\) if and only if it is on any other complete locally convex Suslin space \((E, \tau')\) compatible with duality of \((E, \tau)\).

**Proof.** As both topologies \(\tau\) and \(\tau'\) are locally reducible if and only if the weak topology is locally reducible, the assertion follows from Theorem 7.4.10.

7.4.2 Summary and Examples

For convenience we summarize some classes of spaces which are separably extendable:

1. Banach and Fréchet spaces (directly by Proposition 7.4.1)

2. Nuclear duals of nuclear spaces (by Example 7.4.2)

3. All \(K_0\)-co-Schwartz spaces, thus all \(K_0\)-conuclear, co-Schwartz and all conuclear spaces, thus all metrizable nuclear spaces (by Corollary 7.4.6)
The following examples are spaces on which all infinitely divisible distributions are locally reducible.

1. Separable Fréchet and Banach spaces (by Example 7.3.13)
2. Regular strict inductive limits of a sequence of Fréchet spaces (by Example 7.3.13)
3. Spaces with a finer locally convex Suslin topology \( \tau' \) such that every \( \mu \in \mathcal{I}(E) \) is locally reducible (by Theorem 7.4.10)
4. Spaces possessing a comparable Suslin topology \( \tau' \) compatible with duality such that every \( \mu \in \mathcal{I}(E) \) is locally reducible (by Theorem 7.4.10)

The following spaces are locally convex Suslin spaces with reducing properties: The spaces of distributions and test functions \( \mathcal{D}, \mathcal{D}', \mathcal{E}, \mathcal{E}', \mathcal{S}, \mathcal{S}' \), spaces of multiplication and convolution operators and their dual spaces \( \mathcal{O}_M, \mathcal{O}'_M \) and \( \mathcal{O}_C, \mathcal{O}'_C \), and the countable product \( \mathbb{R}^\mathbb{N} \) as they are \( \mathcal{K}_0 \)-conuclear, cf. [Sch73, p. 233] and Suslin [Sch73, p. 115-117]. For precise definitions and relations of these spaces see [Hor66, pp. 439-440]. \( L^p, L^p_{\text{loc}} \) and \( \mathcal{C}_c(\mathbb{R}^n) \) (the space of continuous functions with uniform convergence on compacts) are separable Fréchet spaces and therefore \( \mathcal{K}_0 \)-co-Schwartz Suslin spaces.

**Open Questions**

We have the following implications:

\[ E \text{ is separably extendable} \]

\[ \text{Thm 7.3.19} \]

\[ \text{Lemma 7.3.12} \]

\[ \text{E is locally separable} \]

\[ \text{E is locally reducible} \]

\[ \text{Lemma 7.3.5} \]

**Question 7.4.12** (Characterization of reducing properties). It is not known if local separability implies local reducibility for all Lévy measures. Also, it is not clear if local separability implies any of the other two properties. Deeper analysis on the interplay of the structure of locally convex Suslin spaces and Lévy measures is necessary to answer these questions.

**Assumption 7.4.13.** If not explicitly stated differently, we assume for the rest of the thesis that \( P_{X_1} = \mu_1 \) is locally reducible.

Assuming local reducibility, the Banach space \( E_K \) serves as a Banach support of the generalized Poisson exponential of \( \nu|_K \). In this case, well-established results for a stochastic integral in Banach spaces can be applied which is carried out in Section 7.6.
7.5 Cádlág functions in Suslin spaces

Before we begin our investigations on random measures we present some results for cádlág functions with values in Suslin spaces. The results are similar to those in [Bil99, Chapter 3]. We denote by \( \mathcal{D}(T; E) \) the space of càdlàg functions \( \xi : T \to E \). This space will always be endowed with the \( \sigma \)-algebra \( \mathcal{F}_\mathcal{D} \) of cylinder sets on \( \mathcal{D}(T; E) \) which are generated by the \emph{coordinate functions} \( x_t : \mathcal{D}(T; E) \to E \) defined by \( x_t(\xi) := \xi(t) \), \( t \in T \). For \( t \in T \) the \emph{left limit mapping} \( x_{t-} : \mathcal{D}(T; E) \to E \) is defined by \( x_{t-}(\xi) := \lim_{s \uparrow t} x_s(\xi) \) and the \emph{jump function} by \( \Delta \xi_t := x_t(\xi) - x_{t-}(\xi) \), \( t \in T \).

Given a dense subset \( T_0 \subseteq T \), càdlàg functions are defined as follows: \( \xi : T_0 \to E \) is càdlàg if and only if for all increasing or decreasing Cauchy sequences \( (t_n)_{n \in \mathbb{N}} \) in \( T_0 \) the limits of \( \xi(t_n) \) exist and if \( t_n \searrow t \in T_0 \) the limit equals \( \xi(t) \). As above, \( \mathcal{D}(T_0; E) \) denotes the set of such càdlàg functions. \( \xi \in \mathcal{D}(T_0; E) \) is said to have a jump in \( s \in T \), if the limits

\[
y_s := \lim_{r \searrow s \in T_0} \xi(r) \quad \text{and} \quad y_{s-} := \lim_{r \nearrow s \in T_0} \xi(r)
\]

are different. \( \Delta \xi(s) := y_s - y_{s-} \) is the jump size in \( s \) and \( s \mapsto \Delta \xi(s) \) is the jump function corresponding to \( \xi \).

**Lemma 7.5.1.** Let \( T_0 \subseteq [0, t_{\text{max}}] \) be dense. For a càdlàg function \( \xi \in \mathcal{D}(T_0; E) \), a continuous seminorm \( p \) and \( \varepsilon > 0 \) there exist finitely many points \( 0 = t_0 < t_1 < \ldots < t_n = t_{\text{max}} \) such that

\[
\sup \{ p(\xi(t) - \xi(s)) : s, t \in [t_{i-1}, t_i) \cap T_0 \} < \varepsilon, \quad i = 1, \ldots, n. \tag{7.5.1}
\]

Furthermore, for any weaker metric \( d \) on \( E \), there are finitely many points with

\[
\sup \{ d(\xi(t), \xi(s)) : s, t \in [t_{i-1}, t_i) \cap T_0 \} < \varepsilon, \quad i = 1, \ldots, n.
\]

**Proof.** Let \( \pi : E \to E/p^{-1}(\{0\}) \) be the canonical projection associated to \( p \) which is continuous. Furthermore, \( \pi \circ \xi \in \mathcal{D}(T; E/p^{-1}(\{0\})) \) and all the limits exist by completeness of \( E \). The target space is normed, but not necessarily complete. The expression in (7.5.1) remains the same if the càdlàg function is projected onto the quotient space. Exactly as in [Bil99, Chapter 3, Lemma 1, p. 110], where completeness is not needed, one obtains the first assertion.

If \( d \) is a continuous metric on \( E \), \( \text{id} \circ (E, \tau) \to (E, d) \) is continuous and by the same reasoning as above the assertion follows.

**Lemma 7.5.2.** A càdlàg function with values in a complete locally convex Suslin space has at most countably many jumps.
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Proof. Let \((E, \tau)\) be the Suslin space with the original topology and \((E, d)\) a metric space, where \(d\) is a weaker metric. Let \(\text{id}: (E, \tau) \rightarrow (E, d)\) be the continuous identity map. If \(\xi \in D(T_0; E)\) has a jump in \(t_0 \in T\) then \(\text{id} \circ \xi\) has a jump in \(t_0\). Thus, \(\text{id} \circ \xi\) has at least as many jumps as \(\xi\) and is càdlàg as well by continuity of the identity map. Furthermore, if \(\xi\) is continuous in \(t_0\), this carries over to \(\text{id} \circ \xi\), thus the jumps of \(\xi\) and \(\text{id} \circ \xi\) are the same. But for càdlàg functions with values in metric spaces it is well-known that there are at most countably many jumps [EK86, Lemma 4.5.1].

Given \(\xi \in D(T; E)\) one can number the jumps of \(\xi\) in the following way: Choose the (or any other weaker) metric \(d\) of the proof above and measure the jumps as above by \(d(\xi_t, \xi_{t-})\). Then, by Lemma 7.5.1, there are only finitely many jumps on bounded intervals if \(d(\xi_t, \xi_{t-})\) is larger than 1. Therefore, one can denote these jump times by \(t_{1,1}(\xi), t_{1,2}(\xi), \ldots\) (setting \(t_{1,k+1}(\xi) = t_{1,k+2}(\xi) = \ldots = \infty\) if there are \(k\) jumps larger than 1) and obtain thus a numbering. If \(d(\xi_t, \xi_{t-}) \in (\frac{1}{n+1}, \frac{1}{n}]\) for some \(n \geq 1\), we can do the same procedure using \(t_{n,1}(\xi), t_{n,2}(\xi), \ldots\) and so on.

**Lemma 7.5.3.** Given \(n\) and \(j\), the map \(t_{n,j}: D(T; E) \rightarrow T \cup \{\infty\}\) is \(\mathcal{F}_D\)-measurable.

The proof is exactly the same as in [Sat99, Proof of Lemma 20.9] replacing the modulus by \(d\) which is a continuous metric on \(E\).

**Lemma 7.5.4.** For \(t \in T, t > 0\), the left limit mapping \(x_{t-}: D(T; E) \rightarrow E\) is \(\mathcal{F}_D - \mathcal{B}(E)\)-measurable.

Proof. Let \(d\) be a weaker metric on \(E\). We note that \(\mathcal{B}(E, \tau) = \mathcal{B}(E, d)\) and the Borel sets are generated by open \(d\)-balls \(B_x(\varepsilon) := \{y \in E: d(x, y) < \varepsilon\}\), \(x \in E\) due to \((E, d)\) being a separable metric space and therefore second countable. Furthermore, the \(\sigma\)-algebra \(\mathcal{F}_D\) on \(D(T; (E, \tau))\) only depends on \(\mathcal{B}(E)\) and not the precise topology generating it. We define \(x_{t-}^d(\xi) := d - \lim_{s \nearrow t} x_s(\xi)\) for \(\xi \in D(T; (E, \tau))\) and show that \(x_{t-}^d(\xi) = x_{t-}(\xi)\). But this follows from \((E, d)\) being Hausdorff and the continuity of the injection \((E, \tau) \hookrightarrow (E, d)\) and therefore

\[
\begin{align*}
x_{t-}^d(\xi) &= \lim_{s \nearrow t} x_s(\xi) = d - \lim_{s \nearrow t} x_s(\xi) = x_{t-}^d(\xi) \quad \text{for} \ \xi \in D(T; (E, \tau)).
\end{align*}
\]

We test measurability on a generator of \(\mathcal{B}(E)\) and obtain

\[
(x_{t-})^{-1}(B_x(\varepsilon)) = (x_{t-}^d)^{-1}(B_x(\varepsilon)) = \{\xi \in D(T; (E, \tau)): \exists n \in \mathbb{N}, \forall q \in \mathbb{Q} \cap (0, 1/n): x_{t-q}(\xi) \in B_x(\varepsilon)\} \in \mathcal{F}_D,
\]
which proves the lemma.

**Lemma 7.5.5.** The map \(x: D(T; E) \times T \rightarrow E\) defined by \((\xi, t) \mapsto x_t(\xi)\) is \(\mathcal{F}_D \otimes \mathcal{B}(T) - \mathcal{B}(E)\)-measurable.
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Proof. For all \( t \in T \) the map \( x_t : D(T; E) \to E \) is \( \mathcal{F}_D \otimes \mathcal{B} \)-measurable. Using right-continuity of \( \xi \) in the \( d \)-topology we obtain that the \( \mathcal{F}_D \otimes \mathcal{B}(T) \)-measurable maps

\[
x^{(n)}_s(\xi) := \begin{cases} \sum_{k=0}^{2^n-1} x_{t_{\max}(k+1)2^{-n}}(\xi) \mathbb{I}_{\left(\frac{t_{\max}k}{2^n}, \frac{t_{\max}(k+1)}{2^n}\right]}(s), & \text{if } s \in T = [0, t_{\max}] \\
\end{cases}
\]

or

\[
x^{(n)}_s(\xi) := \begin{cases} \sum_{k=0}^{4^n-1} x_{(k+1)2^{-n}}(\xi) \mathbb{I}_{\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]}(s), & \text{if } s \in T = [0, \infty),
\end{cases}
\]

respectively, converge to \( x_t(\xi) \) in the \( \tau \)- and the \( d \)-topology as \( n \) tends to infinity. The latter convergence yields the desired measurability of the joint map. \( \square \)

**Lemma 7.5.6.** The map \( \pi : D(T; E) \times T \to E \) defined by \( (\xi, t) \mapsto x_t(\xi) \) is \( \mathcal{F}_D \otimes \mathcal{B}(T) \)-measurable.

Proof. The proof is exactly the same as in Lemma 7.5.5 taking the left endpoints \( x_{t_{\max}k2^{-n}} \) resp. \( x_{k2^{-n}} \) in the approximating sums. \( \square \)

### 7.6 Jump processes and random measures

Let \( X \) be a Lévy process in \( E \) and \( \nu \in \mathcal{M}(E) \) the Lévy measure of \( \mu_1 = P_X \) and locally reducible. Let \( K \in \mathcal{K}_0^* \) be a \( \nu \)-reducing set. In particular, the properties of Proposition 7.3.16 are valid. The main results of this section will be Theorem 7.6.5 and Proposition 7.6.12.

#### 7.6.1 A constructed process

We begin with some general facts about Poisson random measures, which can be found in [Sat99, Chapters 19 and 20]. Given a \( \sigma \)-finite measure space \((\Theta, \mathcal{B}, \rho)\), one can define a Poisson random measure \( \{N'(B), B \in \mathcal{B}\} \) on some probability space \((\Omega', \mathcal{F}', P')\) with intensity measure \( \rho \). For now, we choose this measure space to be the finite measure space \((T \times E, \mathcal{B}(T \times E), \lambda \otimes \nu|_{K^\infty})\). Then, for \( \omega \in \Omega'_0 \subset \mathcal{F}' \) with \( P'(\Omega'_0) = 1 \), the measure \( N'(\cdot, \omega) \) is supported on a finite number of points of mass 1 on all bounded intervals, and \( N'(\{s \times E, \omega\}) \) has values in \( \{0, 1\} \) for all \( s \in T \), cf. [Sat99, Lemma 20.1].

**Definition 7.6.1 (Poisson integral).** Let \( B \in \mathcal{B}(E) \). The **Poisson integral** with respect to \( N' \) is defined by

\[
Y_t(B)(\omega) := \int_{(0,t] \times B} x \, dN'(s, x)(\omega), \quad \omega \in \Omega'_0
\]

(7.6.1)

and \( Y_t(\omega) := 0 \) for \( \omega \in \Omega'_0 \). If \( B = E \), we write \( Y_t := Y_t(E) \).
Proposition 7.6.2.  (1) For \( t \in T, B \in \mathcal{B}(E) \), the random variable \( Y_t(B) \) defined in (7.6.1) is finite.

(2) \( (Y_t(B))_t \) is a càdlàg Lévy process in \( E \) and \( \mathbb{P}_{Y_t(B)} \) has distribution \( e(t\nu|_{K^c \cap B}) \).
In particular, its characteristics are \((0, 0, t\nu|_{K^c \cap B}, K)\).

(3) \( N'(B)(\omega) = \# \{ s : (s, \Delta Y_s(\omega)) \in B \setminus \{0\} \} \)

Proof. (1) follows from the property that \( N'(-, \omega) \) is supported on a finite number of points for \( \omega \in \Omega'_0 \).

(2) This essentially follows from [Sat99, Proposition 19.5]. For convenience, we adapt the situation to ours: The functional \( a \in E' \) is measurable. We consider the real-valued random variable

\[
Y_t^a(B)(\omega) := \langle Y_t(B)(\omega), a \rangle = \int_{[0,t] \times B} \langle x, a \rangle \, dN'(s, x)(\omega),
\]

which has Fourier transform

\[
\mathbb{E}e^{izY_t^a(B)} = \exp \left( t \int_E \left( e^{iz(x,a)} - 1 \right) \, d\nu|_{K^c \cap B}(x) \right)
= \exp \left( \int_{[0,t] \times E} \left( e^{iz(x,a)1_B(x)} - 1 \right) \, d\lambda \otimes \nu|_{K^c}(s, x) \right), \quad z \in \mathbb{R}.
\]

Since this holds for arbitrary \( a \in E' \), the \( E \)-valued random variable \( Y_t(B) \) has distribution \( e(t\nu|_{K^c \cap B}) \) on \( E \). Independent increments follow from \( N' \) being independently scattered, stationary increments from the structure of the intensity measure. The map \( t \mapsto e(t\nu|_{K^c \cap B}) = \mathbb{P}_{Y_t(B)} \) is a convolution semigroup. Weak continuity of the semigroup of distributions of \( Y_t \) follows from

\[
\left| e(t\nu|_{K^c}) - \delta_0 \right|(f) \leq \left| e^{-t\nu|_{K^c}(E)} \sum_{n=1}^\infty \frac{t^n\nu|_{K^c}(E)^n M^n}{n!} \right|
= e^{-t\nu|_{K^c}(E)} \left[ e^{t\nu|_{K^c}(E)M} - 1 \right] \to 0 \text{ for } t \searrow 0,
\]

where \( f \in C_b(E) \) and \( \text{im}(f) \subseteq [-M, M] \). Weak continuity of \( \mathbb{P}_{Y_t(B)} \) is shown analogously substituting \( K^c \) by \( K^c \cap B \) in the previous calculation.

(3) If \( s \in T \) such that \( N'(\{s\} \times E, \omega) = 0 \), then \( \Delta Y_s(\omega) = 0 \). If there is an \( x \in B \setminus \{0\} \) such that \( N'(\{(s, x)\}, \omega) = 1 \), then \( \Delta Y_s(\omega) = x \).

For \( A \in \mathcal{B}(T \times E) \) satisfying \( A \subseteq T \times (K \setminus \varepsilon K) \) for some \( \varepsilon \in (0, 1) \) define the Bochner

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\[ \int_A x \, d(\lambda \otimes \nu)(t, x). \]  

(7.6.2)

Indeed, \((\lambda \otimes \nu)|_A\) is finite and concentrated on (a subset of) \(T \times K\). The map \(\psi: T \times K \to E_K, \ (t, x) \mapsto x\), satisfies the conditions of Bochner integrability: It takes values in a Banach space \(E_K \subseteq E\) and is Bochner measurable (which is equivalent to being measurable in separable Banach spaces); for Bochner integrals in locally convex spaces cf. [Tho75, Definition 5, p. 75]. Therefore, (7.6.2) can be considered as a Bochner integral in \(E_K\) or in \(E\).

The measure \(\lambda \otimes \nu\) is \(\sigma\)-finite on \(B(T \times E)\) using the partition \(E = \bigcup_n C_n\) with \(C_0 = K^c\) and \(C_n := \frac{1}{n+1} K \setminus \frac{1}{n+2} K\) for \(n = 1, 2, \ldots\). Indeed, \(\nu(C_n) < \infty\) by Proposition 7.3.16 (ii).

By \(\sigma\)-finiteness of \(\lambda \otimes \nu\), one can construct a Poisson random measure \(N'\) on a probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) with intensity measure \(\lambda \otimes \nu\), cf. [Sat99, Proposition 19.4].

Without loss of generality, \(N'\) from above is just the restriction of this new Poisson random measure on \(T \times E\) to \(T \times K^c\).

Let \(B \in \mathcal{B}(E), B \subseteq C_n\) and \(t \in T\). Analogously as in (7.6.1) the Lévy process

\[ Y_t(B)(\omega) := \int_{[0,t] \times B} x \, dN'(s, x)(\omega) \]

is finite on bounded intervals for almost all \(\omega \in \Omega', \) (otherwise we set the trajectory to zero). It is contained in \(E_K\) for \(n \neq 0\) and Poisson distributed on \(E\) and \(E_K\) by Proposition 7.3.15.

**Definition 7.6.3 (Compensated Poisson integral).** Let \(B \in \mathcal{B}(E)\) with \(B \subseteq C_n\) and \(t \in T\). The **compensated Poisson integral** is defined by

\[ J'(\{0, t\} \times B) := \int_{[0,t] \times B} x \, d\tilde{N}'(t, x) := \int_{[0,t] \times B} x \, dN'(t, x) - \int_{[0,t] \times B} x \, d(\lambda \otimes \nu)(t, x). \]

**Definition 7.6.4 (Uniform convergence on bounded intervals).** Let \(Z^n: T \times \Omega \to E, n = 1, 2, \ldots\) be càdlàg stochastic processes. They are said to **converge almost surely uniformly on bounded intervals of** \(T\), if there exists a set \(\Omega_0 \in \mathcal{F}\) of measure one, such that for all \(\omega \in \Omega_0\), for all seminorms \(p\) generating the topology \(\tau\) on \(E\) and all bounded intervals \(T_0 \subseteq T\) one has that

\[ \sup_{t \in T_0} p(Z^n_t(\omega) - Z^m_t(\omega)) < \varepsilon \]

for every \(\varepsilon > 0\) and \(n, m \in \mathbb{N}\) sufficiently large.

The following theorem covers the main result of this section.
**Theorem 7.6.5** (Convergence of the compensated Poisson integral). With the notation from above, for \( t \in T \), the series

\[
J'_t := \sum_{n=1}^{\infty} J'_t([0, t] \times C_n)
\]

(7.6.3)

is a series of independent random variables in \( E_K \) and converges almost surely in \( E_K \) and \( E \). The convergence is uniform in \( t \) on bounded intervals of \( T \) in \( E_K \) and \( E \). Finally, \( \{J'_t\}_{t \in T} \) is a càdlàg Lévy process in \( E \) with characteristics \((0, 0, \nu|_K, K)\).

**Remark 7.6.6.** We understand the process \( J'_t \) in the following sense: Let \( \Omega'_2 \in \mathcal{F}' \) be a set of measure one such that (7.6.3) converges uniformly on bounded intervals for all \( \omega \in \Omega'_2 \). Then, for \( \omega \in \Omega'_2 \), the process \( J'_t(\omega) \) takes the value of the right-hand side for all \( t \in T \) and \( J'_t(\omega) := 0 \) for \( \omega \in \Omega'_2 \).

**Proof.** Define \( x_n := -\int_{C_n} x \, d\nu(x) \in E_K \subseteq E \). The summands \( J'([0, 1] \times C_n) \) have distributions \( e(\nu|_{C_n}) \ast \delta_{x_n} \) on \( E \) or, by restriction \( e(\nu|_{C_n}) \|_{E_K} \ast \delta_{x_n} \in \mathcal{M}^1(E_K) \). Their sums converge weakly to \( \tilde{e}(\nu|_K) \|_{E_K} \in \mathcal{M}^1(E) \), cf. proof of [Hey10, Theorem 3.4.9].

By the same arguments as in [Det83, Proof of Theorem 2.1], convergence in distribution of the partial sums of the independent random variables yields a.s. convergence in \( E_K \) (cf. [Hey10, Theorem 3.1.6]). Similarly, it converges for every \( t \in T \). \( \{J'_t\}_{t \in T} \) is a Lévy process in \( E_K \) with characteristics \((0, 0, \nu|_K) \|_{E_K}, K)\). Indeed, the independent and stationary increments follow from the definition of \( N' \). The semigroup of distributions equals \( t \mapsto \tilde{e}(t(\nu|_K) \|_{E_K}) \) and is weakly continuous, cf. [Hey10, Theorem 2.3.9].

By Lemma 7.3.3 all random elements on \( E_K \) are also measurable with respect to \( \mathcal{B}(E_K) \). As \( K \) is bounded and closed, the injection \( \iota: E_K \hookrightarrow E \) is continuous and thus the series converges a.s. in \( E \). In order to establish uniform convergence on bounded intervals of \( T \), we fix \( t \in T \), \( t > 0 \) and again use the injection \( \iota \). Continuity implies \( p(\iota(x)) \leq c_p \|x\|_K \) for all \( x \in E_K \) and all seminorms \( p \) generating \( \tau \) and suitable constants \( c_p > 0 \). Therefore,

\[
\sup_{s \in [0, t]} p \left( \iota \left( J'_s - \sum_{n=1}^{N} J'([0, s] \times C_n) \right) \right) \leq c_p \sup_{s \in [0, t]} \left\| J'_s - \sum_{n=1}^{N} J'([0, s] \times C_n) \right\|_K
\]

where the right hand side converges in probability by [Det83, Proof of Thm. 2.1] and due to the fact that \( J' \) is a Lévy process in \( E_K \).

Applying [Sat99, Lemma 20.4] (which can be proven in exactly the same way for Banach spaces) yields a.s. uniform convergence on bounded intervals of the sequence of processes in \( E_K \), a càdlàg limiting process \( J' \) in \( E_K \) as \( \mathcal{D}([0, t]; E_K) \) is closed under uniform convergence. The inequality yields uniform convergence in \( E \) and càdlàg functions in \( E \) as well injecting the \( \mathcal{D}([0, t]; E_K) \) functions into \( \mathcal{D}([0, t]; E) \) by virtue of \( \xi \mapsto \iota \circ \xi \). It remains to state that \( J' \) is a Lévy process in \( E \). Continuity of \( \iota \)

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implies weak continuity of the semigroup \( t \mapsto \mathcal{E}(t\nu|_K) \), the null extension of the family of distributions on \( E_K \).

For the given compact set \( K \subseteq E \) we write
\[
J'_t := \int_{(0,t] \times K} x \, d\mathcal{N}'(s,x) := \sum_{n=1}^{\infty} J'(t \times C_n).
\]

(7.6.4)

**Remark 7.6.7** (Choice of a certain generalized Poisson exponential). As mentioned above, the generalized Poisson exponential \( \mathcal{E}(\nu|_K) \) is unique up to a convolution. From now on we will use a certain representative given by the construction above in order to avoid ambiguities: We define \( \mathcal{E}(\nu|_K) \) as the weak limit of \( \mathcal{E}(\nu|_{C_n}) \ast \delta_{x_n} \).

Given an infinitely divisible distribution \( \nu \) with characteristics \((\gamma, Q, 0, K)\) on a locally convex Suslin space, where \( Q : E' \to E \) is a covariance operator associated to a Gaussian measure \( \nu \), one can define a Wiener process \( W'_t \) on a probability space \((\Omega', \mathcal{F}', P')\) such that \( W'_t \sim \nu \ast t \), cf. [FdLP95, Proposition 5]. The stochastic process \( (W'_t) \) is continuous, has values in \( E \) and
\[
\mathbb{E}(W'_t - \gamma t, a)\langle W'_s - \gamma s, b \rangle = \min\{s,t\}\langle Qa, b \rangle.
\]

For details, we refer to Proposition 7.7.4.

**Proposition 7.6.8.** Let \( X'_t = J'_t + L'_t + W'_t \), \( t \in T \), be defined on a completed probability space \((\Omega', \mathcal{F}', \mathbb{P}')\) where \( L'_t := Y'_t(K^c) \) is defined as in (7.6.1), and let the summands be constructed in a way such that they are independent. Then,

1. \( (X'_t)_{t \in T} \) is a càdlàg Lévy process and
2. \( X'_t \) has characteristics \((\gamma, Q, \nu, K)\).
3. The random measure \( \mathcal{N}' \) satisfies \( \mathcal{N}'(B) = \# \{ s : (s, \Delta X'_s) \in B \times \{0\} \} \) for \( B \in \mathcal{B}(T \times E) \).

**Proof.** (1) \( X'_t \) is càdlàg being a finite sum of càdlàg processes. The independent and stationary increments properties carry over by grouping of independent random variables.

(2) One obtains the characteristics by convolution of the ingredients and noting that \( \mathcal{E}(\nu_1) \ast e(\nu_2) = \mathcal{E}(\nu_1 + \nu_2) \) (as can seen using the Fourier transform) for a Lévy measure \( \nu_1 \) and \( \nu_2 \in \mathcal{M}^b(E) \). In our situation, \( \nu_1 = \nu|_K \), \( \nu_2 = \nu|_{K^c} \).

(3) This property holds for each \( \varepsilon > 0 \) if \( B \cap (T \times \varepsilon K) = \emptyset \). Observing
\[
B \setminus \{0\} = \bigcup_{n=0}^{\infty} (T \times C_n) \cap B
\]

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with $C_n$, $n \in \mathbb{N}^*$, as defined above and $C_0 := E \setminus K$, one has

$$N'(B) = \sum_{n=0}^{\infty} N'((T \times C_n) \cap B), \quad \text{a.s.}$$

and the assertion follows.

### 7.6.2 Definitions on the original space

This section is following the ideas of Sato in [Sat99, Section 20] for the proof of the Lévy-Itô decomposition of finite dimensional Lévy processes. Necessary extra considerations for measurability issues have been carried out in lemmas 7.5.4 and 7.5.5 in the previous section.

**Definition 7.6.9.** Let $B \in \mathcal{B}(T \times E)$ and $X = (X_t)_{t \in T}$ the (original) Lévy process given on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\xi \in \mathcal{D}(T; E)$ be a càdlàg function, $x_t : \mathcal{D}(T; E) \to E$ the coordinate mapping, $x_t^- : \mathcal{D}(T; E) \to E$ the left limit at time $t \in T$ and $\Delta x_t(\xi) := x_t(\xi) - x_t^-(\xi)$. For $\omega \in \Omega$ resp. $\xi \in \mathcal{D}(T; E)$ define

$$N(B, \omega) := \# \{ s : (s, \Delta X_s(\omega)) \in B \setminus \{0\} \} \quad \text{resp.}$$

$$n(B, \xi) := \# \{ s : (s, \Delta x_t(\xi)) \in B \setminus \{0\} \}$$

and for all sets $B$ with $\lambda \otimes \nu(B) < \infty$ set

$$\tilde{N}(B, \omega) := N(B, \omega) - (\lambda \otimes \nu)(B) \quad (7.6.5)$$

$$\tilde{n}(B, \xi) := n(B, \xi) - (\lambda \otimes \nu)(B).$$

$N$ is called the Poisson random measure associated to $X$ and $\tilde{N}$ the compensated Poisson random measure corresponding to $X$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the original probability space and $(\Omega', \mathcal{F}', \mathbb{P}')$ the probability space of the constructed process of the previous section. Both processes $X$ and $X'$ consist of càdlàg paths only by assumption resp. construction. We define the following maps to the space of càdlàg functions:

$$\psi : \Omega \to \mathcal{D}(T; E), \quad \psi(\omega) := X(\omega),$$

$$\psi' : \Omega' \to \mathcal{D}(T; E), \quad \psi'(\omega) := X'(\omega).$$

One obtains $\mathbb{P} \circ \psi^{-1} = \mathbb{P}' \circ (\psi')^{-1}$, where this measure, call it $\mathbb{P}^\mathcal{D}$, is defined on $\mathcal{F}_\mathcal{D}$. This is due to equality of distributions of $X_t'$ and $X_t$ on $E$ and therefore of $X'$ and $X$ on $\mathcal{D}(T; E)$.

The following lemma shows that $\lambda \otimes \nu$ is the compensator of $N$ and $n$. 

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Lemma 7.6.10. Let $B \in \mathcal{B}(T \times E)$ with $\lambda \otimes \nu(B) < \infty$. Then,

$$\mathbb{P}^N(B) = \mathbb{P}'^N(B) = \mathbb{P}^D_n(B).$$

In particular, $N$ and $n$ are a Poisson random measures with intensity measure $\lambda \otimes \nu$.

Proof. We follow the proof of Sato, cf. [Sat99, p. 132]. We have that $n(B, \psi(\omega)) = N(B, \omega)$ and $n(B, \psi'(\omega)) = N'(B, \omega)$ and furthermore they are equal in law, provided that they are $\mathcal{F}_\omega$-measurable. But this follows from lemmas 7.5.3 and 7.5.4 and the fact that $x_{t_{n,j}(\xi)}(\omega)$ and $x_{t_{n,j}(\xi)-}(\omega)$ are $\mathcal{F}_\omega$-measurable due to lemmas 7.5.5 and 7.5.6, thus

$$G(n,j) := \{ \xi \in \mathcal{D}(T; E) : t_{n,j} < \infty \} \in \mathcal{F}_\omega.$$ 

Noting that $n(B, \xi)$ can be written as the $n,j$-series over indicator functions of $G(n,j)$ yields the assertion. \hfill \Box

For a Borel set $A \in \mathcal{B}(E)$ let $\Omega_0$ be the intersection of all subsets $\Omega_0^n$ of $\Omega$ of full measure such that $N(\{0,n\} \times A, \omega) < \infty$, $n \in \mathbb{N}$, if $T = [0, \infty)$. In the case $T = [0, t_{\max}]$, the set $\Omega_0$ consists of all $\omega \in \Omega$ such that $N([0,t_{\max}] \times A, \omega) < \infty$. For $\omega \in \Omega_0$ define

$$X_t(A)(\omega) := \sum_{0 \leq s \leq t} \Delta X_s(\omega) \mathbb{1}_{A \setminus \{0\}}(\Delta X_s(\omega))$$

and if $\omega \in \Omega_0'$ we set the trajectory $X_\cdot(A)(\omega)$ to zero. Furthermore, carrying out the same construction with $n$ instead of $N$, one can define

$$x_t(A)(\xi) := \sum_{0 \leq s \leq t} \Delta x_s(\xi) \mathbb{1}_{A \setminus \{0\}}(\Delta x_s(\xi)),$$

for $\xi \in \mathcal{D}_0$ of full measure $\mathbb{P}^D$ and also $X'_t(A)(\omega)$ on a set $\Omega'_0 \in \mathcal{F}'$ with $\mathbb{P}'(\Omega'_0) = 1$ setting the whole trajectories to zero on the complements $\mathcal{D}_0'$ and $\Omega'_0$.

Then, for all $\omega \in \Omega$ and $\omega' \in \Omega'$, it holds that

$$X_t(A)(\omega) = x_t(A)(\psi(\omega)) \quad \text{and} \quad X'_t(A)(\omega') = x_t(A)(\psi'(\omega')).$$

Letting

$$L_t := X_t(K^c) \quad \text{and} \quad l_t := x_t(K^c)$$

we have

$$L_t(\omega) = l_t(\psi(\omega)) \quad \text{and} \quad L'_t(\omega) = l_t(\psi'(\omega)) \quad \text{for all } \omega \in \Omega, \omega' \in \Omega'.$$
For all \( \omega \in \Omega \) and \( x \in \mathcal{D}(T, E) \) one has

\[
L_t(\omega) = \int_{[0,t] \times K^c} x \, dN(s,x)(\omega) \quad \text{and} \quad l_t(\xi) = \int_{(0,t] \times K^c} x \, dn(s,x)(\xi)
\]

and \( L_t \) and \( l_t \) are càdlàg Lévy processes with characteristics \((0, \nu|_{K^c}, K)\).

**Proof.** From \( \lambda \otimes \nu([0,t] \times K^c) < \infty \) we deduce \( N([0,t] \times K^c, \omega) < \infty \) for \( \omega \in \Omega_1' \) for some \( \Omega_1' \in \mathcal{F} \) with \( \mathbb{P}(\Omega_1) = 1 \) by Lemma 7.6.10. For such \( \omega \in \Omega_1' \), there are \( s_1(\omega), \ldots, s_n(\omega) \in [0,t] \) with \( N(\{s_k(\omega)\} \times K^c, \omega) = 1 \) and for all other \( s \in [0,t] \) one has that \( N(\{s\} \times K^c, \omega) = 0 \). Therefore one obtains

\[
\int_{[0,t] \times K^c} x \, dN(s,x)(\omega) = \sum_{k=1}^{n(\omega)} \Delta X_{s_k}(\omega) = \sum_{s \in [0,t]} \Delta X_s(\omega) \mathbb{1}_{K^c}(\Delta X_s(\omega)).
\]

If

\[
\omega \in \Omega_1 := \bigcap_{n=1}^{\infty} \Omega_1^n,
\]

the above expression exists for every \( t \in T = [0,\infty) \). If \( T = [0,t_{\max}] \), \( \Omega_1 := \Omega_{1_{\max}} \). By (7.6.8) the same follows for \( l_t \) taking \( \xi \in \mathcal{D}_1 \) with \( \mathbb{P}^\xi \)-measure one. The distributions of \( L_t, l_t \) and \( L_1' \) are the same by (7.6.8). So the weakly continuous semigroup of distributions of \( L_1' \) carries over to \( L_t \) and \( l_t \). Therefore, the finite-dimensional distributions on the path space of \((L_t)_{t \in T}, (l_t)_{t \in T}\) and \((L_1')_{t \in T}\) coincide which implies that the processes \( L_t \) and \( l_t \) have independent and stationary increments. Furthermore, \( L_t \) is càdlàg by construction.

Also the compensated integral with respect to \( N \) resp. \( n \) can be constructed analogously as above by setting

\[
J([0,t] \times C_n) := \int_{[0,t] \times C_n} x \, dN(s,x) - \int_{[0,t] \times C_n} x \, d(\lambda \otimes \nu)(s,x) \quad \text{resp.}
\]

\[
j([0,t] \times C_n) := \int_{[0,t] \times C_n} x \, dn(s,x) - \int_{[0,t] \times C_n} x \, d(\lambda \otimes \nu)(s,x).
\]

Again,

\[
J([0,t] \times C_n)(\omega) = j([0,t] \times C_n)(\psi(\omega)) \quad \text{and}
\]

\[
j'( [0,t] \times C_n)(\omega') = j([0,t] \times C_n)(\psi'(\omega'))
\]

for all \( \omega \in \Omega \) resp. \( \omega' \in \Omega' \).

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Proposition 7.6.12. Defining

\[ J_t := \int_{(0,t] \times K} x \, d\tilde{N}(s,x) := \sum_{n=1}^{\infty} J([0,t] \times C_n) \quad \text{resp.} \quad \tag{7.6.9} \]

\[ j_t := \int_{(0,t] \times K} x \, d\tilde{n}(s,x) := \sum_{n=1}^{\infty} j([0,t] \times C_n) \quad \tag{7.6.10} \]

one has the following:

1. The series (7.6.9) resp. (7.6.10) converge almost surely in \( E_K \) (thus in \( E \)) uniformly in \( t \) on bounded intervals.

2. \( J_t(\omega) = j_t(\psi(\omega)) \) and \( J'_t(\omega') = j_t(\psi'(\omega')) \) for almost all \( \omega \in \Omega \) resp. \( \omega \in \Omega' \).

3. \( J_t \) and \( j_t \) are càdlàg Lévy process with characteristics \((0,0,\nu|_{K},K)\).

Proof. Analogously to Lemma 7.6.11, one obtains that the processes

\[ J^n_t := \sum_{k=1}^{n} J([0,t] \times C_k), \quad j^n_t := \sum_{k=1}^{n} j([0,t] \times C_k), \quad \text{and} \quad (J'_t)^n := \sum_{k=1}^{n} J'([0,t] \times C_k) \]

are equal in distribution, where it has been proved above that the distributions of \((J'_t)^n, n = 1,2,\ldots \) converge to an infinitely divisible measure with characteristics \((0,0,\nu|_{K},K)\) for \( n \to \infty \) and \( t \in T \).

Let \( \Omega'_2 \) be the set of full \( \mathbb{P}' \)-measure such that the series (7.6.3) for \( J'_t \) converges in Theorem 7.6.5. It equals the intersection over all sets \( \Omega'_2(N) \) of \( \mathbb{P}' \)-measure one \((N = 1,2,\ldots)\) where

\[ \Omega'_2(N) := \left\{ \omega \in \Omega' : \lim_{n \to \infty} \sup_{j,k \geq n} \sup_{t \in [0,N]} \| (J'_t)^j(\omega) - (J'_t)^k(\omega) \|_K = 0 \right\}. \]

Let \( \mathcal{D}_2 \) resp. \( \Omega_2 \) be the intersection of sets \( \mathcal{D}_2(N) \) resp. \( \Omega_2(N) \) from above with \( \Omega' \) replaced by \( \mathcal{D}(T;E) \) resp. \( \Omega \) and \( J' \) by \( j \) and \( J \), respectively. Then,

\[ 1 = \mathbb{P}'(\Omega'_2) = \mathbb{P}'(\mathcal{D}_2) = \mathbb{P}(\Omega_2) \]

as it only depends on the distribution of the processes. This yields a.s. uniform convergence on bounded intervals of \( j_t \) and \( J_t \) in \( E_K \) and consequently in \( E \).

2. For all \( \omega \in \Omega'_2 \) and all \( n \in \mathbb{N} \) we have \((J'_t)^n(\omega) = j^n_t(\psi(\omega)) \). As the left-hand side and therefore the right-hand side converges uniformly we obtain that \( \psi(\omega) \in \mathcal{D}_2 \) and \( j^n_t(\psi(\omega)) \to j_t(\psi(\omega)) \) uniformly in \( t \) on bounded intervals by definition of \( \mathcal{D}_2 \) and therefore \( J'_t(\omega) = j(\psi'(\omega)) \) for \( \omega \in \Omega'_2 \). The same arguments hold for \( J \) on \( \Omega \).
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(3) $J$ is càdlàg by uniform convergence and $J$ and $j$ are Lévy processes by equality of their finite dimensional distributions with those of $J'$.

Finally, we set $Y_t(\omega) := X_t(\omega) - L_t(\omega) - J_t(\omega)$ and $y_t(\xi) := x_t(\xi) - l_t(\xi) - j_t(\xi)$ and $Y'_t(\omega) := X'_t(\omega) - L'_t(\omega) - J'_t(\omega) = W'_t(\omega) + \gamma t$.

**Proposition 7.6.13.** The three processes $(J_t, L_t, Y_t)_{t \in T}$ are independent.

**Proof.** Using the correspondences via $\psi$ and $\psi'$ we get for all $B_1, B_2, B_3 \in B(E)$ that

$$
\mathbb{P}((J_t, L_t, Y_t) \in B_1 \times B_2 \times B_3) = \mathbb{P}((j_t \circ \psi, l_t \circ \psi, y_t \circ \psi) \in B_1 \times B_2 \times B_3) \\
= \mathbb{P}'((j'_t \circ \psi', l'_t \circ \psi', y'_t \circ \psi') \in B_1 \times B_2 \times B_3) \\
= \mathbb{P}((J'_t, L'_t, Y'_t) \in B_1 \times B_2 \times B_3) \\
= \mathbb{P}'(J'_t \in B_1)\mathbb{P}'(L'_t \in B_2)\mathbb{P}'(Y'_t \in B_3) \\
= \mathbb{P}(J_t \in B_1)\mathbb{P}(L_t \in B_2)\mathbb{P}(Y_t \in B_3).
$$

The same can be carried out for all finite sets of time instances $t_1 < \ldots < t_n$ which yields independence.

7.7 Wiener processes in locally convex spaces

7.7.1 Characterization of Wiener processes

We consider Gaussian random variables with values in a locally convex space $E$. Probably the most comprehensive monograph concerning Gaussian measures on locally convex topological vector spaces is [Bog91], the following concepts are taken mainly from chapters 2 and 3. See also e.g. [Bil99, Maa09].

A random variable $X: \Omega \to E$ is called Gaussian, if for all $a \in E'$ the real-valued random variables $\langle X, a \rangle = a(X)$ are Gaussian. Let $\varrho$ be a Gaussian measure. A mapping $q: E \to \mathbb{R}_+$ is a $\varrho$-measurable seminorm if there exists a $\mathcal{B}(E)$-measurable linear subspace $E_0 \subseteq E$ with $\varrho(E_0) = 1$ and such that the restriction $q|_{E_0}$ is a seminorm on $E_0$. Obviously, $\| \cdot \|_K$ is a $\varrho$-measurable seminorm if $\varrho(E_K) = 1$.

**Lemma 7.7.1** (Zero-one law for Gaussian measures, [Bog91, 2.5.5]). Let $E_0 \subseteq E$ be a $\mathcal{B}(E)$-measurable affine subspace and $\varrho$ a Gaussian measure. Then, $\varrho(E_0) \in \{0, 1\}$.

For later use, we formulate a proposition about Gaussian measures which can be reduced to a separable Banach subspace.

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Proposition 7.7.2 (Zero-one law and moments). Let $\varrho$ be a centered Gaussian measure on a locally convex Suslin space $E$. If there exists a set $K \in K_0(E)$ of positive measure, one has

1. $\varrho(E_K) = 1$,
2. $\varrho$ is a Gaussian measure on $E_K$ and
3. there exists $\alpha > 0$ such that $\int_{E_K} e^{\alpha\|x\|^2_K} \, d\varrho(x) < \infty$.

In particular, on a separably extendable Suslin space every Gaussian measure has a Banach support.

The assertions follow from the above cited zero-one law, Proposition 7.3.15 and by Fernique’s theorem for $\varrho$-measurable seminorms, [Bog91, Theorem 2.8.5]. The last assertion guarantees the existence of all moments of Gaussian random variables, whenever there exists a compact set of positive measure. But this follows from $E$ being a Suslin, thus a Radon space.

Let $w_t$ be a real-valued standard Brownian motion, $\sigma \geq 0$ and $\gamma \in \mathbb{R}$. We consider every real-valued process of the form $y_t = \sigma w_t + \gamma t$ a one-dimensional Brownian motion.

Definition 7.7.3. A Wiener process with values in $E$ is a continuous Lévy process.

Proposition 7.7.4. For a stochastic process $W = (W_t)_{t \in T}$ with values in $E$ the following assertions are equivalent:

1. $W$ is a Wiener process.
2. $W$ is a continuous process with independent increments such that $(W_t, a)_t$ is a (possibly degenerate) one-dimensional Brownian motion for all $a \in E'$.
3. $W$ is a càdlàg Lévy process with Gaussian distributed increments.
4. $W$ is a càdlàg Gaussian process determined by the mean $\gamma \in E$ (i.e. $\gamma$ satisfies $\mathbb{E}(W_t, a) = (\gamma, a)$ for all $a \in E'$) and a symmetric, positive semidefinite operator $Q \in \mathcal{L}(E', E)$, where $E'$ is equipped with the Mackey topology $\tau_\mu(E', E)$, such that

$$\mathbb{E}(W_t - \gamma t, a)(W_s - \gamma s, b) = (Qa, b) \min\{s, t\}$$

for all $s, t \in T$ and $a, b \in E'$.

Proof. (1) $\implies$ (2): If $W$ is a continuous Lévy process, the same holds for $(W, a)$ and it is well-known that a continuous Lévy process in $\mathbb{R}$ is a Brownian motion with drift of the form $\sigma_a w_t + \gamma_a t$ with $\sigma_a \geq 0$ and $\gamma_a \in \mathbb{R}$ and $w = (w_t)_{t \in T}$ is a real-valued standard Brownian motion.
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(2) \Rightarrow (3): If $W$ is continuous, it is càdlàg. For the stationary increments property we note that the distribution of $W_t - W_s$ equals the distribution of $W_{t-s}$ on the $\pi$-system of cylindrical sets, thus on $\mathcal{E}(E) = \mathcal{B}(E)$. Furthermore, for all $a \in E'$ we have that $\mathbb{P}(W,a)$ is real-valued Gaussian which means that $W_t$ is Gaussian distributed for all $t$. Gaussian distributed increments follow from stationarity.

(3) \Rightarrow (4) If $W_t$ obeys a Gaussian law, there exists an element $\gamma_t \in E$ and a continuous operator $Q_t : E' \to E$ (where $E'$ is equipped with the Mackey topology) such that $\langle \gamma_t, a \rangle = \mathbb{E}(W_t, a)$ and $\langle Q_t a, b \rangle = \mathbb{E}(W_t - \gamma_t, a)\langle W_t - \gamma_t, b \rangle$ for all $a, b \in E'$ by [Bog91, Lemma 3.2.1].

For simplicity, assume for the following that $\gamma = 0$. The independent increments property yields

$$\mathbb{E}(W_t, a)\langle W_s, b \rangle = \mathbb{E}(W_t - W_s, a)\langle W_s, b \rangle + \mathbb{E}(W_s, a)\langle W_s, b \rangle = \langle Q_s a, b \rangle$$

and one obtains the equality

$$\langle Q_t a, b \rangle = n\langle Q_{t/n} a, b \rangle$$

for every $n \in \mathbb{N}$ and $t \in T$ by writing $W_t$ as a telescoping sum over an equidistant time net and using independent and stationary increments. Finally, one obtains $Q_t = tQ_1 = tQ$ by $W_t$ being càdlàg and using

$$\langle Q_t a, a \rangle = \lim_{s \in \mathcal{Q}, s \searrow t} \mathbb{E}(W_s, a)(W_s, a) = \lim_{s \in \mathcal{Q}, s \searrow t} s\langle Q_s a, a \rangle = t\langle Q_s a, a \rangle$$

and polarization. Similarly one obtains $\gamma_t = t\gamma_1 = t\gamma$.

(4) \Rightarrow (3): It suffices to check independent and stationary increments of the given process, but this follows immediately by the covariance structure of the occurring Gaussian vectors.

(3) \Rightarrow (1): I. In the sequel, let

$$S_\mathcal{Q} := \begin{cases} (S \cap \mathcal{Q}) \cup \{ \max S \} & \text{max } S \text{ exists,} \\ S \cap \mathcal{Q} & \text{else} \end{cases}$$

for some interval $S \subseteq [0, \infty)$. Let $\varrho_t := \mathbb{P}(W_t)$ and $H \subseteq E$ be an absolutely convex compact set with $\varrho_1(H) > 0$. Without loss of generality we assume that $\mathbb{E}W_t = 0$ for all $t$, i.e. all Gaussian distributions are centered. Then, the zero-one law for Gaussian measures, cf. [Bog91, Theorem 2.5.5.], implies that $\varrho_1(E_H) = 1$, where $E_H$ is the linear hull of $H$. Furthermore, for every $m \in \mathbb{N}$, one has

$$\varrho_m(E_H) = \varrho_1^m(E_H) = \varrho_1^m(E_H + \ldots + E_H) \geq \varrho_1(E_H)^m \geq 1,$$

which implies together with Lemma 7.7.5 below that $\varrho_t(E_H) = 1$ for all $t \in T$. The
compact sets \( n \cdot H \subseteq E \) are metrizable, [Bog91, Propositions A.1.7 and A.3.16] and we denote by \( d_n \) a metric inducing the same topology on \( n \cdot H \). This can be achieved, e.g., taking a weak metric \( d \) and restrict it to the subspaces \( n \cdot H \).

We take \( T = [0, t_0] \), \( t_0 > 0 \). If the claim holds on all bounded intervals, it is true on \( \mathbb{R}_+ \). First, let us prove that \( \mathbb{P}(W_t \in E_H, t \in T) = 1 \). As \( \mathbb{P}(W_t \in E_H) = \varrho_t(E_H) = 1 \) for every \( t \in T \), one has \( \mathbb{P}(W_t \in E_H : t \in T_\varrho) = 1 \). For an \( \omega \) in this set of full measure we consider the trajectory \( t \mapsto W_t(\omega) \). We follow [Bog91, proof of Proposition 7.2.3] and show that there exists an \( n(\omega) \in \mathbb{N} \) such that \( \{W_t(\omega) : t \in T_\varrho\} \subseteq n(\omega) \cdot H \). To this end, we define the (possibly infinity-valued) process \( \eta_t := \|W_t\|_H \). Let \( d \) be a metric on \( E \) inducing a weaker topology. Defining the continuous function \( Q \) the set \( \{W_t(\omega) : t \in T_\varrho\} \) is compact and absolutely convex in \( E \). If the claim holds on all bounded intervals, it is true on \( \mathbb{R}_+ \).

Due to Doob’s inequality and Fernique’s theorem (cf. [Bog91, Theorem 2.8.5]) one has

\[
\mathbb{E} \sup_{t \in T_\varrho} \eta_t^2 \leq 2 \mathbb{E} \eta_0^2 = 2 \int_E \|x\|_H^2 d\varrho_0(x) < \infty
\]

which implies for \( \omega \in \Omega_0 \), a set of measure one, that \( \sup_{t \in T_\varrho} \|W_t(\omega)\|_H < \infty \). But this yields the existence of an \( n(\omega) \in \mathbb{N} \) with \( W_t(\omega) \in n(\omega) \cdot H \) for all \( t \in T_\varrho \). The limit

\[
W_t(\omega) = \lim_{s \downarrow t} W_s(\omega)
\]

is an element of the compact set \( n(\omega) \cdot H \) by completeness. It follows that \( \{W_t(\omega) : t \in T\} \subseteq n(\omega) \cdot H \leq E_H \) for \( \omega \) in a set of measure one.

II. The second part of the proof follows an idea of [GY15, Proposition A.1]. In [FdLP95, Proposition 5] a probability space \( (\Omega', \mathcal{F}', \mathbb{P}') \) and a process \( (X_t)_{t \in T} \) are constructed such that the latter has the same finite-dimensional distributions as \( W \) and continuous trajectories in \( E \). As above, it is argued that they are almost surely in \( E_H \), where \( H \in K_0(E) \) with \( \varrho_0(H) > 0 \). Let

\[
\Omega'_0 := \{\omega \in \Omega' : \{X_t(\omega) : t \in [0, t_0]_\mathbb{Q}\} \subseteq E_H\} \quad \text{and} \quad \Omega'_n := \{\omega \in \Omega' : \{X_t(\omega) : t \in [0, t_0]_\mathbb{Q}\} \subseteq n \cdot H\}, \quad n = 1, 2, \ldots
\]

The latter sets form an ascending chain \( \Omega'_n \subseteq \Omega'_{n+1} \) and their union equals \( \Omega'_0 \). By hypothesis, the map \( t \mapsto X_t(\omega) \) is continuous for \( \omega \in \Omega'_0 \) and it is uniformly continuous as a mapping from \( [0, t_0] \) to \( n(\omega) \cdot H \). In particular, its restriction to \( [0, t_0]_\mathbb{Q} \) is uniformly
continuous as well. The set
\[
\mathcal{O}_n' := \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{t, s \in [0, t_0]} \left\{ \omega \in \Omega_n' : d_n(X_t(\omega), X_s(\omega)) \leq \frac{1}{k} \right\} \in \mathcal{F}'
\]
equals \Omega_n' and thus \( P' (\mathcal{O}_n') = P' (\Omega_n') \). This property depends only on the distribution of the process. Setting
\[
\Omega_n := \{ \omega \in \Omega : \{ X_t(\omega) : t \in [0, t_0] \} \subseteq n \cdot H \}, \quad n = 1, 2, \ldots
\]
one obtains that
\[
\mathcal{O}_n := \bigcap_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcap_{t, s \in [0, t_0]} \left\{ \omega \in \Omega_n : d_n(W_t(\omega), W_s(\omega)) \leq \frac{1}{k} \right\} \in \mathcal{F}
\]
has the same measure, \( P(\mathcal{O}_n) = P' (\mathcal{O}_n') = P' (\Omega_n') = 1 \), therefore, \( W(\omega) \) is uniformly continuous on \([0, t_0]_{Q} \) for almost all \( \omega \in \Omega \). One can define a unique continuous extension of the uniformly continuous function \( W(\omega) : [0, t_0]_{Q} \to n(\omega) \cdot H \) to the closure \([0, t_0]_{\text{c}} \) of its domain by setting
\[
\overline{W}_t(\omega) := \begin{cases} W_t(\omega), & t \in Q \cap [0, t_0], \text{ and } \omega \in \mathcal{O}_0, \\ \lim_{s \to t} W_s(\omega), & t \in Q^c \cap [0, t_0], \text{ and } \omega \in \mathcal{O}_0, \\ 0, & \text{ else.} \end{cases}
\]
But as
\[
W_{t-}(\omega) = \lim_{s \to t} W_s(\omega) = \lim_{s \to t} \overline{W}_t(\omega) = \lim_{s \to t} W_s(\omega) = W_t(\omega)
\]
for \( \omega \in \mathcal{O}_0 \) and for every \( t \) by the càdlàg property of \( W \), one obtains that \( W \) has already a.s. continuous sample paths. \( \square \)

**Lemma 7.7.5.** Let \( (\varrho_t)_{0 \leq t \leq 1} \) be a semigroup of Gaussian measures on \( E \). Then,

1. \( \varrho_t(H) \geq \varrho_1(H) \) for every measurable absolutely convex set \( H \) and \( t \in [0, 1] \).

2. If \( H \) is bounded and \( \varrho_1(H) > 0 \), then \( \varrho_t(E_H) = 1 \) for all \( t \).
Proof. By the semigroup property we have \( g_t \ast g_{1-t} = g_1 \). Therefore,

\[
 g_1(H) = g_t \ast g_{1-t}(H) = \int_E \int_E \mathbb{1}_H(x + y) \, d\varrho_t(x) \, d\varrho_{1-t}(y) \\
= \int_E \varrho_t(H - y) \, d\varrho_{1-t}(y) \\
\leq \int_E \varrho_t(H) \, d\varrho_{1-t}(y) = g_t(H),
\]

because for a Gaussian measure \( \mu, a \in E \) and every measurable absolutely convex set \( A \) one has \( \mu(A + a) \leq \mu(A) \) by [Bog91, Theorem 2.8.10]. The second assertion immediately follows from (1) and the zero-one law for linear subspaces. \( \square \)

7.7.2 Series representations

The two following results do not contribute for the Lévy-Itô decomposition but will be of essential use for the construction of a random measure in Section 9.2. Let again \( E \) be a complete locally convex Suslin space. Let \( \mu \) be a centered Gaussian measure on \( E \) with corresponding covariance operator \( Q : E' \to E \). The measure \( \mu \) is Radon and thus the Hilbert spaces \( E'_{\mu} := E'_{\mu}^{L^2(\mu)} \) (the reproducing kernel Hilbert space) and \( \mathcal{H}_\mu := Q(E'_{\mu}) \) (the Cameron-Martin space, \( Q \) defined by continuous extension) are separable, cf. [Bog91, Chapters 2 and 3].

Theorem 7.7.6 (Representation of Gaussian rv’s). Let \( Y \) be an \( E \)-valued centered Gaussian random variable with distribution \( \mu \), let \( \mathcal{H}_\mu \) be the corresponding Cameron-Martin space and \( E'_{\mu} \) the reproducing kernel Hilbert space. Let \( Q \) be the covariance operator of \( \mu \), \( (h_n)_{n \in \mathbb{N}} \) a sequence in \( E' \) which is an orthonormal basis of \( E'_{\mu} \) and \( e_n := Qh_n, n \in \mathbb{N} \). Then,

\[
 Y = \sum_{n=1}^{\infty} \langle Y, h_n \rangle e_n \tag{7.7.1}
\]

and the sum converges a.s. in \( E \).

Proof. Note that \( E'_{\mu} \) has an orthonormal basis \( (h_n)_{n \in \mathbb{N}} \subseteq E' \) [Bog91, Corollary 3.2.8] and the sequence \( e_n := Qh_n, n \in \mathbb{N} \), is an orthonormal basis of the Cameron-Martin space \( \mathcal{H}_\mu \).

By independence of the random variables \( \langle Y, h_n \rangle, n = 1, 2, \ldots, \), Theorem 3.5.1 of [Bog91] yields almost sure convergence of the right-hand side in (7.7.1). Therefore, it suffices to show that the series equals \( Y \). We use the following fact [Bog91, Corollary 3.2.12] for \( g \in E'_{\mu} \) and therefore also for linear functionals \( a \in E' \subseteq E'_{\mu} \): It holds...
that

\[ a = \sum_{n=1}^{\infty} a(e_n)h_n \]  \hfill (7.7.2)

where the sum converges in \( E'_\mu \), i.e. in the \( L^2(\mu) \)-norm and by the Two-Series-Theorem of Kolmogorov it converges \( \mu \)-a.s. on \( E \). Using the duality map \( \langle \cdot, \cdot \rangle \) on \( E \times E' \) we have

\[
\langle Y, a \rangle \overset{a.s.}{=} \sum_{n=1}^{\infty} \langle Y, a(Qh_n)h_n \rangle = \sum_{n=1}^{\infty} \langle Y, a(Qh_n)h_n \rangle = \sum_{n=1}^{\infty} \langle Y, h_n \rangle a(e_n) = \sum_{n=1}^{\infty} \langle Y, h_n \rangle e_n, a \]

where the second equality holds almost surely due to \( Y \) being \( \mu \)-distributed. The last equality is a consequence of Theorem 3.5.1 of [Bog91] which guarantees the a.s. convergence of the series in \( E \) and thus \( a \in E' \) can be exchanged with the evaluation map. The assertion follows as this holds for all \( a \in E' \).

**Theorem 7.7.7 (Wiener process).** Suppose \( W = (W_t)_{t \in T} \) is a Wiener process on \( E \) with \( W_1 \sim \mu \) (assume w.l.o.g. that \( 1 \in T \)) and reproducing kernel Hilbert space \( E'_\mu \). Let \( (h_n)_{n \in \mathbb{N}} \subseteq E' \) be an orthonormal basis of \( E'_\mu \). Then,

\[
W_t = \sum_{n \in \mathbb{N}} (W_t, h_n)Qh_n = \sum_{n \in \mathbb{N}} w^n_t e_n \]  \hfill (7.7.3)

converges a.s. in \( E \) for all \( t \) and \( (w^n_t)_{n \in \mathbb{N}} \) is a sequence of independent one-dimensional Brownian motions.

**Proof.** For the existence of the Cameron-Martin space see [Bog91, Proposition 7.2.2]. By Proposition 7.7.4, \( w^n_t := (W_t, h_n) \), \( n \in \mathbb{N} \), is a one-dimensional Brownian motion. For each \( t \) one obtains a centered Gaussian random variable \( w^n_t = (W_t, h_n) \) with covariance \( \langle Q_t h_n, h_n \rangle = E(w^n_t)^2 = t E(w^n_1)^2 = t \langle Q_1 h_n, h_n \rangle \). For \( s \in \mathbb{R} \) the map \( M_s : E \to E \) denotes the scalar multiplication \( x \mapsto xs \). Then, the distribution of \( W_t \)
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equals that of $\sqrt{t}W_1 \sim \mu \circ M_{t^{-1/2}}$, a Gaussian measure, and the series representation of $W_t$ according to Theorem 7.7.6 yields

$$W_t = \sum_{n=1}^{\infty} \langle W_t, \frac{1}{\sqrt{t}} h_n \rangle \sqrt{t}e_n = \sum_{n=1}^{\infty} \langle W_t, h_n \rangle e_n$$

as $(t^{-1/2}h_n)_{n \in \mathbb{N}}$ is an orthonormal basis of the reproducing kernel Hilbert space of $\mu \circ M_{t^{-1/2}}$ and $(tQ)^{-1/2}h_n = t^{1/2}e_n$ is the corresponding basis of the Cameron-Martin space.

### 7.8 LÉVY-ITÔ-DECOMPOSITION

In this section we will obtain the main result of this part, the Lévy-Itô decomposition of sample paths of a Lévy process. Recall that $E$ denotes a complete locally convex Suslin space.

**Theorem 7.8.1 (Lévy-Itô decomposition).** Let $X$ be an $E$-valued Lévy process with characteristics $(\gamma, Q, \nu, K)$ and $\nu$ locally reducible with reducing set $K$. Then there exist an $E$-valued Wiener process $(W_t)_{t \in T}$ with covariance operator $Q$, an independently scattered Poisson random measure $N$ on $T \times E$ with compensator $\lambda \otimes \nu$ and a set $\Omega_0 \in F$ with $P(\Omega_0)$ such that for all $\omega \in \Omega_0$ one has

$$X_t(\omega) = \gamma t + W_t(\omega) + \int_{[0,t] \times K} x \, d\tilde{N}(s,x)(\omega) + \int_{[0,t] \times K^c} x \, dN(s,x)(\omega) \quad (7.8.1)$$

for all $t \in T$. Furthermore, all the summands in (7.8.1) are independent and the convergence of the first integral in the sense of (7.6.9) is a.s. uniform in $t$ on bounded intervals in $E_K$ and $E$. Additionally, the random measure $N$ is given by

$$N(B) = \# \{ (r, \Delta X_r) \in B : r \in T \}, \quad B \in B(T \times E)$$

with $\lambda \otimes \nu(B) < \infty$.

**Proof.** As the distribution $\mu_1$ of $X_1$ is locally reducible with reducing set $K \in K_0(E)$ it holds that $\nu(K^c) < \infty$ and $\nu$ is a Lévy measure on the separable Banach space $E_K$.

Let $L_t$ and $J_t$ be defined as in (7.6.7) and (7.6.9), respectively. By means of the mappings $\psi$ and $\psi'$ from Section 7.6.2, the processes $(X_t, L_t, J_t)_{t \in T}$ and $(X'_t, L'_t, J'_t)_{t \in T}$ have the same finite-dimensional distributions. Therefore, the process $Z = (Z_t)_{t \in T}$ defined by $Z_t := X_t - L_t$ has the same distribution as $X'_t - L'_t = J'_t + W'_t + \gamma t$ and is therefore a Lévy process. The process $(Z_t)_{t \in T}$ has jumps of a size in $K$ and its characteristics are $(\gamma, Q, \nu|_K, K)$, which is obtained by the characteristics of $X'_t - L'_t$. A.s. uniform convergence of the series in (7.6.9) and the fact that for every jump of
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There exists an \( n \in \mathbb{N} \) such that the jump exceeds \( n^{-1} \cdot K \) imply that for almost all \( \omega \in \Omega \) the trajectory \( Y_t(\omega) := Z_t(\omega) - J_t(\omega) \) is continuous as all jumps are erased. The distribution of \( J_t \) equals \( e(t\nu|K) \) on \( E \) by Theorem 7.6.5. Furthermore, \( Y_t \) has characteristics \( (\gamma, Q, 0, K) \), namely the same as those of \( X'_t - L'_t - J'_t \).

In addition, the continuous process \( Y_t \) has stationary and independent increments, thus it is a continuous Lévy process, a Wiener process with drift in \( E \) by Proposition 7.7.4.

The element \( \gamma \in E \) satisfies \( \mathbb{E}\langle Y_1, a \rangle = \langle \gamma, a \rangle \) for all \( a \in E' \). This is nothing else than \( \gamma = \mathbb{E}Y_1 \) in the Pettis sense. Setting \( W_t := Y_t - \gamma t \) we obtain a centered Wiener process \( W_t \).

Independence of the summands follows from Proposition 7.6.13.

We can even strengthen the result and let the process \( X^0_t := \gamma t + W_t + J_t, t \in T \), live in a Banach space \( E_K \). In order to obtain this, we use Lemma 7.3.18.

**Corollary 7.8.2.** Let \( E \) be separably extendable. Then, there exists \( K \in K^0_0 \) such that \( X_t = X^0_t + L_t, t \in T \) (a.s.), and additionally,

1. \( X^0_t \) has values in \( E_K \) for all \( t \in T \) a.s. and
2. \( X^0_t \) is Bochner integrable and square integrable in \( E_K \) for every \( t \in T \).

**Proof.** The representation \( X_t = X^0_t + L_t \) follows from the Lévy-Itô decomposition. Define \( K := K_1 + K_2 + K_3 \), where \( K_1 \in K^0_0 \) is \( \nu \)-reducing, the set \( K_2 \in K^a_0 \) has positive (centered) Gaussian measure (which exists, as there must be \( K' = K \in K_0 \) of positive measure, therefore \( K_2 \supseteq K' \), \( K_2 \in K^a_0 \) by separable extendability of \( E \)) and the whole trajectory of \( W \) stays in \( E_K \) by the same arguments as in the proof of Proposition 7.7.4. The set \( K_3 \) is the absolutely convex hull of \( \gamma \in E \) and \( E_K \), \( \gamma \in K \) is clearly separable. Lemma 7.3.18 yields that \( K \in K^0_0 \). Note that \( K \) is \( \nu \)-reducing by Lemma 7.3.12, \( K \) has positive centered Gaussian measure (and therefore measure one by [Bog91, Theorem 2.5.5]) and \( \gamma \in K \). Therefore, \( X^0_t \) has values in \( E_K \) a.s. and can be actually considered as a process in the Banach space \( E_K \) by analogous arguments as above. (Square) integrability follows from Proposition 7.7.2 and [PARA03, Corollary 3.4].

**Proposition 7.8.3.** Let \( E \) be separably extendable and \((X_t)_{t \in T} \) a Lévy process with values in \( E \) and characteristics \( (\gamma, Q, \nu, K) \). The following assertions are equivalent:

1. There exists a \( t_0 \in T \) such that \( X_{t_0} \) takes values a.s. in a separable Banach space \( E_1 \) with closed unit ball compact in \( E \).
2. There exists a \( t_0 \in T \) such that \( \mathbb{P}_{X_{t_0}} \) has a Banach support \( E_1 \) with closed unit ball compact in \( E \).
(3) For all \( t \in T \) one has \( X_t \in E_1 \) a.s. for a separable Banach space \( E_1 \) with closed unit ball compact in \( E \).

(4) For all \( t \in T \) the distribution \( P_{X_t} \) has a Banach support \( E_1 \) with closed unit ball compact in \( E \).

(5) \( \nu \) has a Banach support \( E_2 \) with closed unit ball compact in \( E \).

Given (5), one can choose \( E_1 = E_K + E_2 + E_3 + \mathbb{R}_\gamma \), where \( E_3 \) is a suitable Banach support of the Gaussian part of \( X \) and \( K \) is \( \nu \)-reducing.

**Proof.** The equivalence of (1) and (2) resp. (3) and (4) and the implication (4) \( \Rightarrow \) (1) are obvious and (2) \( \Rightarrow \) (4) follows from Lemma 7.3.10.

(2) \( \Rightarrow \) (5): If (2) holds, \( P_{X_{t_0}} \) is infinitely divisible on the Banach space \( E_1 \) and thus, there exists a Lévy measure \( \nu' \) on \( E_1 \). Injecting \( E_1 \) into \( E \), finding the null extensions of \( P_{X_{t_0}} \) and \( \nu' \) and using Proposition 7.3.15 and uniqueness of the Lévy measure of an infinitely divisible distribution, one obtains \( \nu'_{t_0} = \nu \) and therefore, \( \nu \) has Banach support \( E_1 \). One can choose \( E_2 = E_1 \) and obtain assertion (5).

(5) \( \Rightarrow \) (2): Let \( K \in K_0^s \) be \( \nu \)-reducing. Then, \( \varrho(\nu) = \varrho(\nu|_K) * \varrho(\nu|_{K'}\gamma) \) has Banach support \( E_K + E_2 \). Let \( K_2 \in K_0^s \) be the closed unit ball of \( E_2 \). The Gaussian part \( \varrho \) has Banach support \( E_3 = E_H \) for some \( H \in K_0^s \) and \( \delta_\gamma \) has Banach support \( \mathbb{R}_\gamma \). Put \( K' := K + K_2 + H + [-1,1] \cdot \gamma \) which is in \( K_0^s \) by Lemma 7.3.18. Then, \( \mu := \varrho(\nu) * \varrho * \delta_\gamma \) has Banach support \( E_1 := E_{K'} = E_K + E_2 + E_3 + \mathbb{R}_\gamma \) and as \( \mu\|_{E_1} \) is infinitely divisible on \( E_1 \), there exists a root \( (\mu\|_{E_1})^{*t_0} = (\mu^{*t_0})\|_{E_1} \) by Lemma 7.3.10. Therefore, \( P_{X_{t_0}} = \mu_{t_0} = \mu^{*t_0} \) also has Banach support \( E_1 \) which is the assertion. \( \square \)
III

Chaos Expansions for Infinite Dimensional Lévy Processes
In the 1930s, N. Wiener set the starting point for the development of a huge variety of different chaos expansions theorems representing $L^2$-random variables by means of a sequence of deterministic chaos kernels with his homogeneous chaos [Wie38]. Multiple integrals with respect to random measures and corresponding chaos expansion theorems for the Gaussian case were developed by K. Itô [Itô51] and later also for the Lévy case [Itô56]. The latter states that for each $F \in L^2(\Omega, \mathcal{F}^X, P)$, where $\mathcal{F}^X$ is the $\sigma$-algebra generated by increments of the real-valued Lévy process, there exists a sequence of deterministic functions $(f_n)_{n=0}^{\infty}$ such that $F$ can be represented as a series

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

of multiple stochastic integrals $I_n$ of the kernels $f_n$. These multiple integrals are constructed from the Lévy process $X$ and the chaos kernels $f_n$ are elements of Hilbert spaces $L_n^2 := (L^2_1) \otimes^a$, $n \in \mathbb{N}$.

But for general vector-valued Lévy processes no such result is known, or at least has not explicitly been formulated yet. One possible explanation could be the completely different nature of the Poissonian and the Gaussian part of a Lévy process. Indeed, for these two separate parts, chaos expansion results are well-known and experience a
wide variety of possible applications in Malliavin calculus [SUV07a, SUV07b, SU08, GL11], approximation theory [GGL13], BSDEs [BL14, GS13, Ste14b, Ste14a], and they are used to prove covariance inequalities and central limit theorems [HPA95, NP09, PSTU10].

Maybe the most intriguing connection of chaos expansions to physics lies in the Fock space structure of the kernel spaces describing a system of an unknown number of particles. The creation and annihilation operators on the Fock space in a physical interpretation add or subtract particles to the system. From a probabilistic point of view, these operators are nothing else than the Skorohod integral and the chaos counterpart of the Malliavin derivative which can be formally introduced on every Fock space structure.

Before we start with the construction, let us present some known results and applications of the chaos expansion.

**Gaussian chaos.**

Nualart [Nua95] uses the notion of a Gaussian isonormal family associated to a Hilbert space (isonormal families have been introduced by Segal, [Seg54]) and develops a theory of Gaussian chaos expansions using Hermite polynomials. This Fourier-Hermite-expansion goes back to Cameron and Martin, cf. [CM47]. The case of multiple integrals with respect to a Gaussian random measure can be embedded into this setting. This is the so-called white-noise case, where $L^2(T)$ serves as associated Hilbert space. This general approach is capable to describe the case of a chaos expansion theorem of a $d$-dimensional Brownian motion, see [Nua95, Example 1.1.2]. Additionally, by means of series representations of infinite-dimensional Wiener processes (Theorem 7.7.7), we will be able to apply this framework to these more general cases as well in Theorem 9.2.2.

**Poissonian chaos.**

L. Wu [Wu87] proved a chaos expansion in the case of a Poisson point process on a locally compact space with diffuse Radon¹ measure (cf. also Nualart and Vives [NV90, Section 5] for the embedding of this example in their general Fock space approach). This has been recently generalized by Last and Penrose [LP11]. They prove a chaos expansion theorem for Poisson point processes on arbitrary $\sigma$-finite measure spaces. In particular, the latter result can serve as a major step in the proof of a chaos expansion in the case of a pure-jump Lévy process if one can translate the setting of the pure-jump Lévy process into the one of point processes and vice versa. In the case of a finite-dimensional Lévy process, this has been proposed by Kallenberg and Szulga, [KS89, Section 6].

¹In the sense of a locally finite measure.
One might also try to use orthogonal polynomials as a more general concept in the Poissonian case analogously to the Gaussian one. In fact, for the Poisson process this was developed by Ogura [Ogu72] with the help of multivariate Charlier polynomials. But already Ogura, and more explicitly, Kailath and Segall [SK76], stated that Itô’s multiple integrals based on Lévy processes cannot always be represented by Charlier polynomials.

**Stochastic distributions by chaos.**

Considering the Malliavin derivative on Wiener space, the subspaces of $k$-time Malliavin differentiable and $p$-integrable random variables, denoted by $\mathbb{D}^{k,2}$, are well-known examples of Sobolev spaces in a stochastic setting and one can define their dual spaces as Sobolev spaces with negative index $k$, cf. [Wat93].

On Gaussian and Poisson spaces the chaos expansion can be expressed in terms of orthogonal polynomials. The $k$-th Malliavin derivative as an operator on the chaos has the domain $\mathbb{D}^{k,2}$ which can be characterized by growth conditions on the coefficients. But also other growth conditions are possible and yield various spaces. We sketch the case of a Gaussian isonormal family $W$ associated to a separable Hilbert space $H$ with orthonormal basis $(e_i)_{i=0}^{\infty}$ (cf. [Nua95, p. 7]). Let $\mathcal{A} := \mathbb{N}_0^{(\mathbb{N})}$ be the set of all finite sequences of integers. An element $\alpha \in \mathcal{A}$ is called **multi-index**. Define $|\alpha| := \sum_{i=0}^{\infty} \alpha_i$ and $\alpha! := \prod_{i=1}^{\infty} \alpha_i!$. If $H_k$ is the $k$-th Hermite polynomial, set

$$H_{\alpha}(x) := \prod_{i=1}^{\infty} H_{\alpha_i}(x_i) \quad \text{for} \quad x \in \mathbb{R}^\mathbb{N}.$$  

This product is finite as almost all indices are 0 and $H_0 = 1$. Given $\alpha \in \mathcal{A}$ we define

$$\Phi_{\alpha} := \prod_{i=1}^{\infty} H_{\alpha_i}(W(e_i)).$$

The chaos expansion theorem [Nua95, Theorem 1.1.1] states that $\mathcal{F} \in L^2(\Omega, \mathcal{F}^W, \mathbb{P})$ can be written as an $L^2$-converging series

$$F = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} b_\alpha \Phi_{\alpha} \right) = \sum_{\alpha \in \mathcal{A}} b_\alpha \Phi_{\alpha},$$

where $b_\alpha \in \mathbb{R}$. On the other hand, given a countable collection of $b_\alpha$, $\alpha \in \mathcal{A}$, one can describe $L^2$-random variables. But this is only possible if they satisfy the growth condition

$$\sum_{\alpha \in \mathcal{A}} b_\alpha^2 \alpha! < \infty.$$
Introducing stronger growth conditions
\[ \sum_{\alpha \in A} b_\alpha^2 q_\alpha < \infty \]
using weights \( q_\alpha > \alpha! \), \( \alpha \in A \), one obtains subclasses of random variables, stochastic test functions, denoted by \( (Q) \subseteq L^2(\Omega, \mathcal{F}^W, \mathbb{P}) \), cf. [Lev11, Section 2.5]. It should be remarked that also smooth random variables in the Malliavin sense can be described by growth conditions on the kernels.

Relaxing the convergence conditions one can define generalized random variables as formal Hermite expansions with growth condition
\[ \sum_{\alpha \in A} c_\alpha^2 q_\alpha < \infty \]
with certain weights \( q_\alpha < 1 \) on the coefficients \( c_\alpha \). These spaces are denoted by \( (Q)^{-} \).

One can define an action of \( F \in (Q)^{-} \) with coefficients \( c_\alpha \) on a stochastic test function \( f \in (Q) \) given by coefficients \( b_\alpha \) by
\[ \langle\langle F, f \rangle\rangle := \sum_{\alpha \in A} \alpha! c_\alpha b_\alpha, \]
which is well-defined if the sum converges absolutely. This depends on the choice of weights \( q_\alpha \) and \( \overline{r}_\alpha \). Examples are given in [Lev11, Section 2.5] obtaining Kondratiev test functions if \( f \) is in the space with weights \( q_\alpha := (\alpha!)^{1+\varrho} r_\alpha^p \) for a certain parameter \( \varrho \in [0,1] \) for all \( p \in \mathbb{N}_0 \) and \( r_\alpha := \prod_{k=1}^{\infty} (2k)^{\alpha_k} \). Kondratiev distributions are formal expansions such there exists a \( p \in \mathbb{N}_0 \) with \( F \) is in the space with weights \( \overline{r}_\alpha := \alpha!^{1-\varrho} r_\alpha^{-p} \) and a given parameter \( \varrho \in [0,1] \). Hida distributions are a subclass taking \( \varrho = 0 \).

A similar approach seems also possible for chaos expansions by means of multiple integrals. Also in this case, one can impose growth conditions on the norms of the chaos kernels. Malliavin fractional smoothness (in the sense of real interpolation) for example can be characterized in this manner, cf. [GGL13, p. 9] and [GH07, A.1].

Allowing negative regularity by means of weak growth conditions, one can again define stochastic distributions. The action of those on test functions leads to a series of inner products of the corresponding symmetric chaos kernels. As it is not possible in the Lévy case to use orthogonal polynomials, one needs a description in terms of multiple integrals. This is a nearly untouched research field and not treated so far in the literature.
Mixed Lévy case.

About the chaos expansion for general Lévy processes in multiple dimensions only little is known and published. Recently, Steinicke suggested in his PhD thesis a random measure for which a chaos expansion theorem holds in the case of $\mathbb{R}^d$-valued Lévy processes, cf. [Ste14a, Section 1.4.3]. However, the result is stated without proof.

In the following, we will present a proof for the general case of a locally reducible Lévy process in a locally convex Suslin space by splitting it into independent parts according to the Lévy-Itô decomposition. For the diffusion part, we will use the classic approach of a Gaussian isonormal family which has to be defined in a suitable way. The jump part will be described by the random measure $N$ associated to the Lévy process $X$ using the chaos expansion results for Poisson point processes. Combining these two completely different approaches, one can find a random measure for the multiple stochastic integrals in this situation.
We recall the constructions of the introductory Section 2.3 going back to Itô, cf. [Itô56] and adapt them to the more general setting. Let \( M : S \times \Omega \to \mathbb{R} \) be an independently scattered random measure (cf. [App09, Section 2.3.1] and [Kal02, p. 106]) with non-atomic \( \sigma \)-finite intensity measure \( m \) on the measurable space \((S, S)\). We define \( L^2_n := L^2(S^n, S^\otimes_n, m^\otimes_n) \) and its subspace \( C^2_n \) of simple functions of the form \( f_n(s_1, \ldots, s_n) = 1_{A_1}(s_1) \cdots 1_{A_n}(s_n) \) with mutually disjoint sets \( A_1, \ldots, A_n \in S \) such that \( m(A_k) < \infty \), \( k = 1, \ldots, n \) and \( s_1, \ldots, s_n \in S \). Itô proves that the linear span of \( C^2_n \) is dense in \( L^2_n \), [Itô51, Theorem 2.1].

**Definition 9.0.1 (Multiple integral).** For \( n = 1, 2, \ldots \) the multiple integral \( I_n : L^2_n \to L^2(\Omega, \mathcal{F}, \mathbb{P}) \) is defined by

\[
I_n(f_n) := M(A_1) \cdots M(A_n)
\]

for functions \( f_n = 1_{A_1} \cdots 1_{A_n} \in C^2_n \) and by linear and continuous extension on \( L^2_n \). If \( n = 0 \), we set \( L^2_0 := C^2_n := \mathbb{R} \) and \( I_0(c) := c \).

The following proposition is proved in the same way as in the one-dimensional case.

**Proposition 9.0.2 (Properties of the multiple integral).** Let \( n, m \in \mathbb{N} \) and \( f_n \in L^2_n, f_m \in L^2_m \). The norm of \( L^2_n \) is denoted by \( \| \cdot \|_n \). The symmetrization of \( f_n \in L^2_n \) is denoted by \( f_n^\circ \), see (2.3.2). Then,

1. \( \mathbb{E}I_n(f_n)I_m(f_m) = \delta_{nm} n! \| f_n^\circ \|^2_n \) and
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\[
  (2) \quad I_n(f_n) = I_n(\tilde{f}_n).
\]

Defining multiple integrals on \((S, S, m)\) is always possible by the above construction. But it has to be proven in the specific situations that one can represent all elements of the target space \(L^2(\Omega, \mathcal{F}, \mathbb{P})\) by a series of multiple integrals of suitable kernel functions in \(L^2_n\). We will define random measures according to pure jump Lévy processes and Wiener processes with values in locally convex Suslin spaces and prove a chaos decomposition result. Finally, the Lévy-Itô decomposition will enable us to use these results for the proof of the chaos decomposition for vector-valued Lévy processes.

Throughout the rest of the chapter, let \(E\) always be a complete locally convex Suslin space and \((X_t)_{t \in T}\) a Lévy process which is locally reducible with reducing set \(K \in \mathcal{K}_0(E)\) and let \(J_t, L_t\) and \(W_t\) be the small jump part, the big jump part and the Wiener process obtained in the Lévy-Itô decomposition, Theorem 7.8.1.

9.1 Chaos expansion for jump processes

In this section, let \(X_t = \gamma t + J_t + L_t\) be a pure-jump Lévy process with values in \(E\). Let \(\mathcal{F}_0' := \sigma(X_t - X_s : s, t \in T, s < t)\) and \(\mathcal{F}^J := \mathcal{F}_0' \vee \mathcal{N}\), where \(\mathcal{N} := \{A \in \mathcal{F} : \mathbb{P}(A) = 0\}\). For the definition of a random measure consider \((S, S, m) := (\mathbb{T} \times E, \mathcal{B}(\mathbb{T} \times E), \lambda \otimes \nu)\), where \(\nu\) is the Lévy measure of \(X\). Denote by \(\mathcal{B}(\mathbb{T} \times E)_0 := \{A \in \mathcal{B}(\mathbb{T} \times E) : \lambda \otimes \nu(A) < \infty\}\) a ring of Borel sets of finite measure. We consider the random measure \(M := N - \lambda \otimes \nu\), the compensated Poisson random measure. Here, \(N : \mathcal{B}(\mathbb{T} \times E)_0 \times \Omega \to \mathbb{N}\) is defined by \(N([0, t] \times B) := \#\{\Delta X_s \in B : 0 \leq s \leq t\}\), it is indeed Poisson distributed for every \(A \in \mathcal{B}(\mathbb{T} \times E)_0\) and Lemma 7.6.10 yields that \(\lambda \otimes \nu\) is the intensity measure of \(\tilde{N}\).

**Theorem 9.1.1** (Chaos decomposition, pure-jump Lévy process). For each \(F \in L^2(\Omega, \mathcal{F}^J, \mathbb{P})\) there exist \(f_n \in L^2_n\), \(n \in \mathbb{N}\), such that

\[
  F = \sum_{n=0}^{\infty} I_n(f_n) \quad \text{a.s.,}
\]

where the series consists of mutually orthogonal summands. Furthermore, we have that

\[
  \mathbb{E}|F|^2 = \sum_{n=0}^{\infty} \mathbb{E}|I_n(f_n)|^2 = \sum_{n=0}^{\infty} \|\tilde{f}_n\|_n^2.
\]

We will use a chaos expansion result of Last and Penrose [LP11] for Poisson point processes in order to obtain our expansion. A Poisson point process is a synonym of Poisson random measure, cf. [Kal02]. Let \((S, S)\) be a measurable space and \(\eta : S \times \Omega \to \mathbb{N}\) a Poisson random measure with intensity measure \(m(A) := \mathbb{E}\eta(A, \cdot)\). We denote
9.1 Jump processes

by $\mathcal{M}_\mathbb{N}(S)$ the integer-valued measures on $(S, \mathcal{S})$ and $\mathcal{B}$ is the $\sigma$-algebra on this space making all mappings $\mu \mapsto \mu(B), B \in \mathcal{S}$, measurable. We can consider $\eta$ as a random element in $\mathcal{M}_\mathbb{N}(S)$ by setting

$$\tilde{\eta}: \Omega \rightarrow \mathcal{M}_\mathbb{N}(S), \quad \omega \mapsto \eta(\cdot, \omega).$$

The result [LP11, Theorem 1.3] of Last and Penrose is as follows. For $f \in L^2(\mathcal{M}_\mathbb{N}(S), \mathcal{B}, P_\eta)$ there exist unique, symmetric and deterministic functions $f_n \in L^2(S, m)$ for $n \in \mathbb{N}$ and multiple integrals $I_n$ with respect to the compensated Poisson random measure $\tilde{\eta} := \eta - m$, such that

$$f(\eta) = \sum_{n=0}^{\infty} I_n(f_n).$$

Let $\mathcal{F}_0 := \sigma(\tilde{\eta}) = \tilde{\eta}^{-1}(\mathcal{B})$. Using a factorization lemma [Bau01, Lemma II.11.7] it is easy to show that every $F \in L^2(\Omega, \mathcal{F}_0, P)$ corresponds uniquely to an $f \in L^2(\mathcal{M}_\mathbb{N}(S), \mathcal{B}, P_\eta)$ by the factorization of $F = f \circ \tilde{\eta}$. The map $F \mapsto f$ is an isometric isomorphism between the two spaces.

In order to prove Theorem 9.1.1 with the help of these results, the following lemma is essential.

**Lemma 9.1.2.** $\mathcal{F}_0^J \vee \mathcal{N} = \mathcal{F}^N \vee \mathcal{N}.$

**Proof.** It is clear that $\mathcal{N}$ is a limit of measurable terms of increments of $X$, thus $N(B)$ and $\tilde{N}(B)$ are $\mathcal{F}^J$-measurable for all $B \in \mathcal{B}(T \times E)$.

For the converse inclusion we need to write $X_t - X_s$ in terms of $N$. By the Lévy-Itô-decomposition we have that

$$X_t - X_s = \gamma(t - s) + (L_t - L_s) + (J_t - J_s)$$

$$= \gamma(t - s) + \int_{(s,t] \times K} x \, dN(r,x) + \int_{(s,t] \times K} x \, d\tilde{N}(r,x) \quad a.s.$$ 

The latter term is the limit of linear combinations of terms in $N$ in the Banach space $E_K$ by the following construction: Let $\varepsilon > 0$ and $E_0 = \{p_1, p_2, \ldots\} \subseteq E_K \setminus \varepsilon \cdot K$ be countable and dense in $E_K \setminus \varepsilon \cdot K$, thus bounded away from zero. Then, for $n \in \mathbb{N}$ and $p \in E_0$ denote by $B^n_p := \{x \in E: \|x - p\|_K < \varepsilon^{-1}\}$. Let $C^n_1 := B^n_p \setminus K$ and iteratively $C^n_k := B^n_{p_k} \setminus (K \cup \bigcup_{\ell=1}^{k-1} C^n_{\ell})$. Omitting all empty sets and applying a new numbering for every $n$ this yields a sequence of nonempty and disjoint sets covering $E_K \setminus \varepsilon K$. For each $k, n$ choose an arbitrary $x^n_k \in C^n_k$. The series

$$P^n_t := \sum_{k=1}^{\infty} x^n_k N([0, t] \times C^n_k)$$
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has a.s. finitely many summands distinct from 0 and for these \( \omega \) one has

\[
\| P^n_t(\omega) - J^n_t(\omega) \|_K \leq \frac{1}{n} N(E_K \setminus \varepsilon K, \omega).
\]

Thus, for every \( t \), \( P^n_t \to J^n_t \) a.s. for \( n \to \infty \). Another a.s. limit letting \( \varepsilon \downarrow 0 \) yields \( J_t \).

For the term of the big jumps one also needs to construct a suitable partition of \( E \) which is done using a weaker metric \( d \) on \( E \) introduced in Section 5. As \( (E, d) \) is Suslin, there is a countable dense subset \( E_1 \subseteq E \). The family \( (B^n_p)_{p \in E_1} \) covers \( E \) for all \( n \in \mathbb{N} \). An analogous construction as above yields disjoint sets \( C^n_k, k \in \mathbb{N} \) and points \( y^n_k \in C^n_k \).

Without loss of generality, \( C^n_k \) can be chosen in way that the intersection with \( E_K \) is empty. Let

\[
Q^n_t := \sum_{k=1}^{\infty} y^n_k N([0, t] \times C^n_k).
\]

As above,

\[
d(Q^n_t(\omega), L_t) \leq \frac{1}{n} N([0, t] \times (E \setminus E_K), \omega)
\]

for almost all \( \omega \), thus \( Q^n_t \to L_t \) a.s. in the \( d \)-topology, and by \( B(E, \tau) = B(E, d) \), the limit is \( \mathcal{F}^N \cup \mathcal{N} \)-measurable.

Now we are in the position to easily prove the main theorem.

**Proof of Theorem 9.1.1.** Let \( F \in L^2(\Omega, \mathcal{F}^N, \mathbb{P}) \) and \( N \) the Poisson random measure on the measure space \( (S, S, m) = (T \times E, B(T \times E), \lambda \otimes \nu) \). Then there exist symmetric functions \( f_n \in L^2((T \times E, B(T \times E), \lambda \otimes \nu)^n) \) and multiple integrals \( I_n \) such that

\[
F = \sum_{n=0}^{\infty} I_n(f_n).
\]

The lemma above shows that \( \mathcal{F}^N \cup \mathcal{N} = \mathcal{F}^J \) and Theorem 9.1.1 is thus proved.

### 9.2 Chaos expansion for Wiener processes

We set \( (S, S, m) := (T \times \mathbb{N}, B(T) \otimes B(\mathbb{N}), \mu) \) with \( \mu = \lambda \otimes \sum_{n \in \mathbb{N}} \delta_n \) and \( L^2_n \) is defined as usual on this measure space. The random measure is defined as \( M([0, t] \times \{i\}) := w^i_t \), see below for details. The approach in this section is based on isonormal Gaussian families, cf. [Nua95]. First we prove a measurability lemma which allows the application of this well-established theory in our context.
**Lemma 9.2.1.** Let $W$ be a Wiener process with values in a Suslin locally convex space $E$, $(h_n)_{n \in \mathbb{N}} \subseteq E'$ be an orthonormal basis of $E'_\rho$ and let $(w^n)_{n \in \mathbb{N}}$ be the corresponding sequence of real Brownian motions such that the series representation (7.7.3) holds. Define the $\sigma$-algebras

$$
F^W := \sigma(W_t - W_s : s < t) \lor \mathcal{N}, \text{ and }
$$

$$
G^W := \sigma(w^n_t - w^n_s : n \in \mathbb{N}, s < t) \lor \mathcal{N}. \tag{9.2.1}$$

Then, $F^W = G^W$.

**Proof.** By definition, $w^n_t = (W_t, h_n)$ and therefore,

$$
F^W = \sigma(W_t - W_s : s < t) \lor \mathcal{N}
$$

$$
\supseteq \sigma(W_t - W_s, h_n : n \in \mathbb{N}, s < t) \lor \mathcal{N}
$$

$$
= \sigma(w^n_t - w^n_s : n \in \mathbb{N}, s < t) \lor \mathcal{N} = G^W.
$$

The inverse inclusion follows from noting that the series converges also in $(E, d)$ where $d$ is a weaker metric on $E$ and by the series representation (7.7.3).

**Theorem 9.2.2 (Chaos decomposition vector-valued Wiener process).** Let $(W_t)_{t \in T}$ be a Wiener process with values in a locally convex Suslin space $E$, let $H_\rho$ be the Cameron-Martin space of $\rho := P_W$, let $(h_n)_{n \in \mathbb{N}}$ be an orthonormal system in $H_\rho$ and let $((w^n_t)_{t \in T})_{n \in \mathbb{N}}$ be independent, real-valued Brownian motions such that (7.7.3) holds. Define

$$
W(\mathbb{1}_{[0,t] \times \{i\}}) := w^n_t.
$$

Then we have that $W$ is an isonormal Gaussian family associated to $L_t^2$ by

$$
\mathbb{E}W(\mathbb{1}_{[0,t] \times \{i\}})W(\mathbb{1}_{[0,s] \times \{j\}}) = (s \wedge t)\delta_{i,j}.
$$

Let $I_n$ denote the $n$-th multiple integral w.r.t the random measure

$$
W(A) := \sum_{n \in J} W(A_n \times \{n\}),
$$

where $A = \bigcup_{n \in J} A_n \times \{n\}$ such that $\mu(A) < \infty$ and $J \subseteq \mathbb{N}$. Then, for all $F \in$
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\( L^2(\Omega, \mathcal{F}, P) \) there exist functions \( f_n \in L^2 \) such that

\[
F = \sum_{n \in \mathbb{N}} I_n(f_n)
\]  

(9.2.3)

**Proof.** This result is immediate by [Nua95, Theorem 1.1.2] and the fact that \( \mathcal{F} = G^W \) which has been proved in the above lemma.

9.3 Chaos expansion for Lévy processes

Let \( X_t: \Omega \to E, t \in T \), be a Lévy process with Wiener process part \( W_t \) and jump parts \( J_t \) and \( L_t \) as in Theorem 7.8.1. Furthermore, let \( \nu \) be the Lévy measure of \( X_1 \).

If \( (U, \mathcal{U}) \) and \( (V, \mathcal{V}) \) are measurable spaces with \( U \) and \( V \) disjoint, one easily checks that the \( \sigma \)-algebra \( U \vee V := \sigma(A \cup B : A \in \mathcal{U}, B \in \mathcal{V}) \) on \( U \vee V \) equals \( UV := \{A \cup B : A \in \mathcal{U}, B \in \mathcal{V}\} \). If two measures \( \mu_1 \) and \( \mu_2 \) are given on \( \mathcal{U} \) and \( \mathcal{V} \), respectively, the prescription

\[
\mu(A \cup B) := \mu_1(A) + \mu_2(B)
\]

is a well-defined measure on \( U \vee V \).

In our context, \( (U, \mathcal{U}) = (\mathbb{N}, 2^\mathbb{N}) \) and \( (V, \mathcal{V}) = (E, \mathcal{B}(E)) \), \( \mu_1 = \sum_{i=1}^{\infty} \delta_i \) and \( \mu_2 = \nu \). Therefore, we can define

\[
\mathfrak{m}([a, b] \times (A \cup B)) := \lambda([a, b]) \otimes \left( \sum_{i=1}^{\infty} \delta_i(A) + \nu(B) \right)
\]

for \( a, b \in T, a < b \) and \( A \in 2^\mathbb{N}, B \in \mathcal{B}(E) \). We set

\[
(S, S, m) := \left(T \times (\mathbb{N} \cup E), \mathcal{B}(T) \otimes (2^\mathbb{N} \vee \mathcal{B}(E)), \mathfrak{m} \right)
\]

and the corresponding random measure on \( (S, S, m) \) is defined by

\[
M([a, b] \times (A \cup B)) := W([a, b] \times A) + \tilde{N}([a, b] \times B),
\]

for \( a, b \in T, a < b \) and \( A \in 2^\mathbb{N}, B \in \mathcal{B}(E) \) such that \( \mathfrak{m}([a, b] \times (A \cup B)) < \infty \). One immediately obtains

**Lemma 9.3.1.** \( M \) is an independently scattered random measure with intensity measure \( \mathfrak{m} \).
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**Theorem 9.3.2** (Chaos decomposition for vector valued Lévy processes). Let $\mathcal{F}_0^X$ denote the $\sigma$-algebra generated by the increments of $X$, let $\mathcal{F}^X := \mathcal{F}_0^X \lor \mathcal{N}$ and $I_n$ be the multiple Wiener integrals defined on $L_n^2$, $n = 0, 1, 2, \ldots$. Then, for all $F \in L^2(\Omega, \mathcal{F}_X, \mathbb{P})$ there exist $f_n \in L_n^2$ such that

$$F = \sum_{n=0}^{\infty} I_n(f_n)$$

and the series converges in $L^2(\Omega, \mathcal{F}_X, \mathbb{P})$.

**Proof.** Analogously as in the real case – see [SUV07a, Lemma 3.1] – we obtain that $\mathcal{F}^X = \mathcal{F}^W \lor \mathcal{F}^J$ by the Lévy-Itô-decomposition: The inclusion $\mathcal{F}^X \subseteq \mathcal{F}^W \lor \mathcal{F}^J$ is a direct consequence, the converse inclusion is proved as follows. In fact, we have $\mathcal{F}^N \lor \mathcal{N} \subseteq \mathcal{F}^X$ (as in the proof of Lemma 9.1.2), where it also has been proved that $L_t$ and $J_t$ are $\mathcal{F}^N \lor \mathcal{N}$-measurable. Writing $W_t := X_t - \gamma t - L_t - J_t$, the Brownian part is $\mathcal{F}^X$-measurable as well.

As $\mathcal{F}^W$ and $\mathcal{F}^J$ are independent (by the Lévy-Itô decomposition) we have $L^2(\Omega, \mathcal{F}^X, \mathbb{P}) \cong L^2(\Omega, \mathcal{F}^W, \mathbb{P}) \otimes_2 L^2(\Omega, \mathcal{F}^J, \mathbb{P})$, (see e.g. [HL01]) and sums of products of $W$ and $\tilde{N}$ are dense, which is exactly obtained by the definition of $M$ as random measure for the multiple integrals. \hfill $\Box$
**Outlook**

At the end of this thesis we give some exemplary questions and research issues emerging from this work.

**Invariances and chaos expansion.**

The investigation of chaos kernel properties in the analysis of random variables seems a promising research field. The approach in this thesis provided a simpler, more tractable chaos expansion for many widely used examples. In view of applications like simulation of BSDEs [BL14] with the help of chaos expansions, a better understanding of the interplay of chaos kernels and corresponding random variables is desirable. Are there other natural properties that can be transferred between chaos kernels and random variables? Is there maybe even a possibility to compute the kernels explicitly?

**Stochastic distributions and chaos expansion.**

As indicated in Chapter 8, chaos expansions can be used to introduce stochastic distributions and Sobolev spaces of negative order. The known constructions make use of expansions in terms of orthogonal polynomials and it would be interesting if, and to what extent one could carry this over to chaos expansions given by multiple integrals.

**Characterization of extendability properties.**

In Section 7.4 sufficient conditions for local reducibility, separable extendability and local separability were formulated. Question 7.4.12 points out that a better understanding of the relations between these notions is desirable, especially as these conditions appear in similar (and in fact, more restrictive) forms in [Det76b, Section 3] and in [Sch73, p. 233, Examples]. The relations need a subtle analysis of the interplay of topology, functional analysis and measure theory.
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Path properties in locally convex Suslin spaces.

The field of (locally convex) Suslin spaces as suitable state spaces for random phenomena is not yet exhaustively researched. Important pioneering work was done by De Wilde [DW69] and Schwartz [Sch73]; vector integration theory and measurability equivalences were introduced and investigated by Thomas and Labuda [Tho75, Val75, Lab78]; a central limit theorem was obtained by Schachermayer [Sch81], a Radon-Nikodým result by Blondia [Blo84]. The emphasis of research in locally convex Suslin spaces (and more general concepts like convex-Suslin spaces) is located in functional analysis, typically proving closed-graph theorems, cf. [Fer90, FKLP09]. If one drops the assumption of $E$ being locally convex, many results in measure theory on topological spaces can be found in Bogachev [Bog07, chapters 6 and 7] and Radon Gaussian measures on locally convex spaces are considered in [Bog91, Chapter 3]. Finally, Schwartz considers Markov processes with values in completely regular Suslin spaces [Sch77]. On the other hand, a law of large numbers was obtained in separable locally convex spaces which are submetirc [CRdF00] and infinitely divisible measures were considered in complete locally convex spaces [Det76a, Tor69].

But a comprehensive theory of random elements and processes with values in locally convex Suslin spaces, their distributions and measurability properties, path properties, structural properties, and so on, is still missing.

As it turns out in this thesis, many measurability arguments have to be obtained by detours to auxiliary metric spaces. However, this requires knowledge about convergence in the original topology which is typically generated by uncountably many seminorms. For example, the following question (the answer of which is affirmative in Polish spaces) appears to need innovative methods for its treatment:

**Question 9.3.3.** Let $X = (X_t)_{t \in T}$ be a stochastic process with values in a complete locally convex Suslin space $E$ and let $Y = (Y_t)_{t \in T}$ be a continuous resp. càdlàg stochastic process such that the finite-dimensional distributions of $X$ and $Y$ coincide. Does there always exist a continuous resp. càdlàg version of $X$?

Subordinated processes.

One research goal is the investigation of specific examples of Lévy processes in locally convex Suslin spaces. A simple method to construct new classes of Lévy processes out of known ones in finite dimension is the procedure of subordination. It is a time change of the given Lévy process $(X_t)_{t \in \mathbb{R}_+}$ by means of an independent, real-valued and increasing Lévy process $Z$ (which is called a subordinator). This means, one defines a new process $Y_t(\omega) := X_{Z_t(\omega)}(\omega)$ for $\omega \in \Omega$, $t \geq 0$. In finite dimensions (cf. [Sat99, Chapter 6] and in the case of a Banach state space of $X$ (cf. [PARA03]) one can describe the distributions of the subordinated process $Y$ by means of the
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characteristics of $X$ and the subordinator $Z$. Is this also true in locally convex Suslin spaces? Is $Y$ always a Lévy process? How do the characteristics – and in particular the reducing set $K$ – transfer to the characteristics of $Y$? Is the new reducing set – if it exists – constructable?

**Stochastic integrals.**

In finite dimensions, one can use the Lévy-Itô decomposition in order to define stochastic integrals with respect to Lévy processes by treating the jump and the diffusion part differently, cf. [App09, Section 4.3.3]. This is also possible in an infinite-dimensional setting, cf. [App07, Equation 5.4] where this is discussed for Banach spaces. [Üst82] and [FM15] develop a stochastic integration theory for certain nuclear spaces taking advantage of the Hilbertian local spaces. For the case of a general locally convex space (or a $K_0$-co-Schwartz space or similar restrictions), a stochastic integral has not been constructed so far. However, if one would like to consider SPDEs in these cases, this is an inevitable step and therefore of high interest. The question is, of course: Is it possible to give a useful integral definition if the underlying Lévy process or the solution process has values in locally convex Suslin spaces?
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