Stability of infinite slopes investigated with elastoplasticity and hypoplasticity

Standsticherheit unendlich langer Böschungen untersucht mit Elastoplastizität und Hypoplastizität

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It is commonly assumed that the maximum slope angle β of an infinite cohesionless slope is equal to the friction angle of the soil φ . However, this relation holds only for a dilatancy angle ψ equal to the friction angle if the Mohr-Coulomb failure criterion is employed. This article demonstrates that the increase of the soil strength due to the plain-strain conditions in an infinite slope allows for using $\beta = \varphi$ as a conservative approximation for $\varphi < 35^\circ$ and $\psi > \varphi/3$. Further, the presented calculations support the common practice in geotechnical engineering of using the strength parameters obtained from triaxial tests in limit state calculations like the Bishop's method without further consideration of the dilatancy. However, it seems to be more appropriate to use direct shear tests when possible, i.e. for fine grained soils.

Es wird üblicherweise angenommen, dass die maximale Neigung β einer unendlich langen, kohäsionslosen Böschung dem Reibungswinkel φ des Materials entspricht. Bei der Verwendung des Versagenskriteriums nach Mohr-Coulomb ist das jedoch nur korrekt für Böden mit Dilatanzwinkel ψ gleich dem Reibungswinkel. In diesem Beitrag wird gezeigt, dass in Folge des ebenen Verzerrungszustandes $\beta = \varphi$ als konservative Schätzung verwendet werden kann wenn $\varphi < 35^\circ$ und $\psi > \varphi/3$ gilt. Außerdem bestätigen die angestellten Berechnungen die übliche Vorgehensweise in der Geotechnik Standsticherheitsuntersuchungen, wie zum Beispiel das Lamellenverfahren nach Bishop, mit Festigkeitsparameter aus dem Triaxialversuch ohne Berücksichtigung der Dilatanz durchzuführen. Es scheint jedoch angemessener, wenn möglich Rahmenscherversuche durchzuführen, um die erforderlichen Parameter zu erhalten.

1. Introduction

The mechanical model of the infinite slope is very simple and hence can be easily implemented into Geographical Information Systems (eg. [1, 2, 3]). This model is also very clear and trains the ability to interpret slope stability problems mechanically.

2. Friction angle and angle of dilatancy of soil

We extensively use the terms friction angle and angle of dilatancy in this article and will therefore start with a definition of our nomenclature. The mobilized friction angle calculated

with, σ'_I and σ'_{III} , the major and the minor principal stress (compression positive), respectively

$$\sin \varphi_m = \frac{\sigma'_I - \sigma'_{III}}{\sigma'_I + \sigma'_{III}} \quad (1)$$

is related to a Mohr-Coulomb failure criterion. This mobilized friction angle reaches during shearing a maximum value, the so-called peak friction angle $\varphi_p = \max \varphi_m$, and (for initially dense samples) will decrease to the so-called critical friction angle φ_c . The specimen will change its volume throughout shearing. An angle of dilatancy can be calculated to be used in a flow rule of an elasto-plastic material model. For example, in triaxial conditions it follows for a Mohr-Coulomb like flow rule

$$\tan \psi_m = -\frac{\dot{\varepsilon}_v}{2\dot{\varepsilon}_1 - \dot{\varepsilon}_v}, \quad (2)$$

where we use the same index m as for the mobilized friction angle, to emphasize that these two values are coupled. We denote the value of ψ_m at the peak of shear strength with ψ_p . At critical state $\psi_m = \psi_c = 0$. Another mobilized friction angle can be calculated from a simple shear or direct shear test with the applied normal stress σ'_n and the measured shear stress τ

$$\tan \phi_m = \frac{\tau}{\sigma'_n}, \quad (3)$$

which is related to a Coulomb failure criterion. Again we will find a peak value for dense specimens $\phi_p = \max \phi_m$ and a critical value ϕ_c for large shear strains. The peak friction angle is used in the calculations following. Hence, we will abbreviate $\varphi := \varphi_p$ and $\psi := \psi_p$ in triaxial tests, as well as $\phi := \phi_p$ in simple or direct shear tests.

3. Mohr-Coulomb – elastoplastic

The text book derivation for the stability of an infinite slope relates the normal stress σ'_n at the bottom of a lamella (Fig. 1(a)) with the shear stress at failure τ_f using a Coulomb failure criterion. With the silent assumption of $\varphi = \phi$ it follows that the inclination of the slope β at limit state is equal to the friction angle φ (e.g.[4]). Teunissen and Spierenburg [5] posed the question if the material strength in the lamella is high enough to remain as a rigid body in this limit state calculation. They employed an ideal plastic material model with

Figure 1: Infinite slope, possible failure of an element.

Abbildung 1: Unendlich lange Böschung, mögliche Versagensarten eines Elementes.

a Mohr-Coulomb failure criterion and non-associated flow rule and found out that $\varphi = \phi$ is only valid for an associated flow rule, which is in line with the collapse theorems of plasticity theory [6]. However, soil does not have a dilatancy angle equal to the friction angle. In triaxial test the dilatancy of Hostun Sand at peak is between $\varphi_p/4$ and $\varphi_p/3$ [7]. Teunissen and Spierenburg [5] proposed a relation for the limit state of a slope, which depends on the friction angle and the dilatancy angle of the material

$$\tan \beta = \frac{\sin \varphi \cos \psi}{1 - \sin \varphi \sin \psi} \quad (4)$$

If the slope failure is assumed to be a simple shear mechanism (Fig. 1 (b)), equation (4) can be derived analytically from a simple shear element test employing a linear-elastic perfectly plastic material model with a Mohr-Coulomb yield function and a non-associated flow rule, see Appendix A. The flow rule (as derivation of a plastic potential with respect to the stress) in such models implies coaxiality of the stress tensor and the plastic stretching tensor. However, calculations with an elasto-plastic material model with a non-coaxial flow rule ([8], [9]) will predict the same values of β . From a relation between Coulomb and Mohr-Coulomb friction angle proposed in [10] for simple shear conditions, it follows for a non-coaxial flow rule (similar to a relationship suggested in [11])

$$\tan \beta = \frac{\sin \varphi \cos(\psi + 2\iota)}{1 - \sin \varphi \sin(\psi + 2\iota)} \quad (5)$$

with $\iota = \alpha_\sigma - \alpha_\epsilon$, being the angle of non-coaxiality, i.e. the deviation of the principal directions of the stress tensor α_σ and the strain rate tensor α_ϵ , compare Appendix C. Numerical simple shear experiments with discrete elements showed for $K = 0.5$ that the angle of non-coaxiality starts with $\iota < 0$ and monotonically increases to $\iota = 0$ (i.e. coaxiality) for large strains. This means that β from (5) is smaller or equal to β from (4), which confirms the computations with the elasto-plastic model mentioned above [8]. For $K = 1$ the angle of non-coaxiality $\iota \approx 0$ throughout the entire shearing [10].

Figure 2: Limit state of a cohesionless infinite slope, β is the inclination, φ the friction angle and ψ the dilatancy angle. $g = 0$ defines the ultimate state (c.f. (6)), the filled area is the state of failure.

Abbildung 2: Grenzzustand einer kohäsionslosen Böschung, β ist die Böschungsneigung, φ der Reibungswinkel und ψ ist der Dilatanzwinkel des Materials.

Eq. (4) is limited to the case of an earth pressure coefficient at rest $K_0 < 1$, which should be applicable for most slopes. The case $K_0 > 1$ is studied in Appendix B.

A limit state function g depending on the friction angle φ , the dilatancy angle ψ and the inclination of the slope β can be established, so that in an ultimate state the value of this function is zero. This limit state function reads

$$g = \tan \beta - \frac{\sin \varphi \cos \psi}{1 - \sin \varphi \sin \psi} \quad (6)$$

for equation (4). For $g(\varphi, \psi, \beta) < 0$ the slope is defined as stable and a state $g(\varphi, \psi, \beta) > 0$ is not feasible, see Fig. 2. The red line defines the ultimate state of the slope ($g(\varphi, \psi, \beta) = 0$). A point above the line $g = 0$ results in a stable slope, whereas a point below (yellow filled area) results in a failure of the slope. In all figures $\varphi = \beta$ is plotted as a reference. In Fig. 3 the limit state functions for several dilatancy angles are shown.

Relation (4) was previously introduced by Davis [12] to incorporate non-associated plasticity in a slip-line analysis. He proposed to use the reduced-strength parameters

$$\tan \varphi^* = \frac{\sin \varphi \cos \psi}{1 - \sin \varphi \sin \psi} \quad (7)$$

$$c^* = c \frac{\cos \varphi \cos \psi}{1 - \sin \varphi \sin \psi} \quad (8)$$

in combination with an associated flow rule. The same relations were proposed by Drescher and Detournay [13] for the use in translational failure mechanisms, when a coaxial flow rule is used. The reduced strength parameters have also been recently

Figure 3: Limit state $g = 0$ of a cohesionless infinite slope in an elastoplastic material model with Mohr-Coulomb yield function, Eq. (6).

Abbildung 3: Grenzzustand $g = 0$ einer kohäsionslosen Böschung mit einem elastoplastischen Materialmodell und einer Mohr-Coloumb Fließfläche, Gl. (6).

introduced in slope stability analysis by means of finite element limit analysis and finite element strength reduction techniques [14, 15, 16]. The slope angle at limit state is then equal to the reduced friction angle: $\beta = \varphi^*$.

The dilatancy angle ψ is zero for critical states and equation (4) yields $\tan \beta = \sin \varphi$. This slope angle is much smaller than the one of the classical approach. Also for other dilatancy angles smaller than the friction angle (4) leads to inclinations β smaller than φ , Fig. 3. Just for the cases $\psi = \varphi$, which means associated flow rule, the inclination at the limit state is the same as the friction angle ($\beta = \varphi$).

However, in an infinite slope a plain-strain condition can be assumed. For this boundary condition it is known, that the Mohr-Coulomb failure criterion leads to conservative solutions, because it does not take for the intermediate principal stress into account.

4. Matsuoka-Nakai – elastoplastic

A more appropriate failure criterion for plain-strain conditions in soils is the Matsuoka-Nakai criterion [17], in which the intermediate principal stress is included

$$\frac{I_1 I_2}{I_3} = k_{MN} \quad , \quad (9)$$

Figure 4: Comparison of Matsuoka-Nakai (blue) and Mohr-Coulomb (red) failure criteria, with marks at triaxial compression test (filled circle), triaxial extension test (circle) and plane strain states (gray plane).

Abbildung 4: Vergleich des Matsuoka-Nakai (blau) und Mohr-Coulomb (rot) Versagenskriteriums, mit Markierungen bei triaxialer Kompression (gefüllter Kreis), triaxialer Extension (Kreis) und bei ebenem Verzerrungszustand (graue Fläche).

where I_1, I_2 and I_3 are the first, second and the third invariants of the stress tensor respectively, which can be expressed as

$$I_1 = \sigma'_I + \sigma'_{II} + \sigma'_{III} \quad (10)$$

$$I_2 = \sigma'_I \sigma'_{II} + \sigma'_{II} \sigma'_{III} + \sigma'_{III} \sigma'_I \quad (11)$$

$$I_3 = \sigma'_I \sigma'_{II} \sigma'_{III} \quad (12)$$

and k_{MN} is a material parameter derived from the friction angle

$$k_{MN} = \frac{9 - \sin^2 \varphi}{1 - \sin^2 \varphi} \quad . \quad (13)$$

An intersection of the failure surface of this criterion with a deviatoric plane is shown in Fig. 4. The material strength is the same as calculated with Mohr-Coulomb in triaxial conditions, whereas higher strength can be mobilised in plane strain conditions $\varphi_m > \varphi$, Fig. 5.

The maximum inclination of a cohesionless slope can be computed from the results of a numerical simple shear test at a constant normal stress σ' and plane strain condition [18]. From this numerical analysis a maximum shear stress τ_f can be found for the applied normal stress. It is reasonable to assume that this stress state is equal to the stress state in an infinite slope at the limit state, i.e.

$$\frac{\tau_f}{\sigma} = \tan \beta \quad . \quad (14)$$

The friction angle used in (13) can be plotted against β [19] cf. Fig. 6.

Comparing Fig. 3 with Fig. 6 it can be concluded that in all cases the maximum resulting slope angles are higher when employing the Matsuoka-Nakai failure criterion instead of the Mohr-Coulomb failure criterion. It is possible to achieve an

Figure 5: End points of simple shear calculations in the principal stress space for different friction angles φ and dilatancy angles ψ .

Abbildung 5: Endpunkte der Simple Shear Berechnungen im Hauptspannungsraum für unterschiedliche Reibungswinkel φ und Dilatanzwinkel ψ .

inclination, which is greater than the friction angle of the material, e.g. $\varphi < 25^\circ$ with $\psi = 0$ and $\varphi < 32^\circ$ with $\psi = \varphi/4$

The results of the calculation with Matsuoka-Nakai criterion can be approximated with

$$\tan \beta = \frac{\sin(1.085\varphi) \cos \psi}{1 - \sin \varphi \sin \psi}, \quad (15)$$

which can be considered as an extension of (4). The results of (15) agree quite well with the results of the numerical simple shear tests in a range from $\psi = 0$ up to $\psi = \varphi/3$ (max. $\Delta\varphi = \pm 5\%$), Fig. 7.

5. Matsuoka-Nakai – kinematic

An different approach towards computing the limit state of a slope is to determine the required friction angle for its stability. For this purpose, in the first step the full stress tensor has to be calculated. The shear and normal stresses in every depth of the slope are already known from the classical approach. The remaining stresses need to be determined. Haefeli [20] has proposed a graphical kinematic solution for this task, an analytical method is presented in [21]. Both methods are based on the assumption that the stress and stretching tensors are

Figure 6: Limit state $g = 0$ for Matsuoka-Nakai failure criterion.

Abbildung 6: Grenzzustand $g = 0$ für das Matsuoka-Nakai Versagenskriterium.

coaxial. The principal stresses obtained in [20, 21] are

$$\sigma'_I = \gamma t \cos \beta \frac{\cos(\beta - \psi) + \sin \beta}{\cos \psi}, \quad (16)$$

$$\sigma'_{II} = \gamma t \cos \beta \frac{\cos(\beta - \psi) + \sin \beta \sin \psi}{\cos \psi}, \quad (17)$$

$$\sigma'_{III} = \gamma t \cos \beta \frac{\cos(\beta - \psi) - \sin \beta}{\cos \psi}. \quad (18)$$

$$(19)$$

With the principal stresses, the invariants of the stress tensor and k_{MN} (cf. (9)) of the Matsuoka-Nakai criterion can be calculated,

$$k_{MN} = \frac{(\sigma'_I + \sigma'_{II} + \sigma'_{III})(\sigma'_I \sigma'_{II} + \sigma'_{II} \sigma'_{III} + \sigma'_{III} \sigma'_I)}{\sigma'_I \sigma'_{II} \sigma'_{III}}. \quad (20)$$

The required friction angle of the material follows from (13)

$$\sin \varphi = \sqrt{\frac{k_{MN} - 9}{k_{MN} - 1}}. \quad (21)$$

For the special case of critical state ($\psi = 0$) this can be simplified to [21]

$$\sin \varphi_c = \frac{\sqrt{18 \cos^2 \beta - 15 \cos^4 \beta - 3}}{5 \cos^2 \beta - 1}. \quad (22)$$

The results of the computation of the required friction angle for different dilatancies are shown in Fig. 8., which agree quite well with the elastoplastic solution with Matsuoka-Nakai.

Figure 9: Triaxial test with Hostun Sand and initial conditions $e_0 = 0.627$, $\sigma'_0 = -100$ kPa.

Abbildung 9: Triaxialversuch mit Hostun Sand und den Anfangsbedingung $e_0 = 0.627$, $\sigma'_0 = -100$ kPa.

Figure 7: Comparison of limit states $g = 0$ with the approximated relation (15) and the elastoplastic model with the Matsuoka-Nakai criterion.

Abbildung 7: Vergleich zwischen den Grenzzuständen mit der angenäherten Beziehung (15) und dem Matsuoka-Nakai Versagenskriterium.

Figure 8: Results of the elastoplastic computation with the kinematic approach.

Abbildung 8: Ergebnisse der elastoplastischen Berechnung mit kinematischem Ansatz.

Figure 10: Simple shear test with Hostun sand and initial conditions $e_0 = 0.627$, $\sigma'_y = -100$ kPa and $\sigma'_x = \sigma'_z = 48.50$ kPa.

Abbildung 10: Simple Shear Berechnung mit Hostun Sand und den Anfangszustand $e_0 = 0.627$, $\sigma'_y = -100$ kPa and $\sigma'_x = \sigma'_z = 48.50$ kPa.

6. Hypoplasticity

Numerical simple shear computations with the hypoplastic material model [22, 23] in particular the version of von Wolffersdorff [24] can be used to assess the limit state in an infinite slope. Triaxial test calculations with the material parameters summarised in Tab. 1 were conducted, e.g. Fig. 9. The friction angle and the dilatancy angle at peak are controlled by the initial isotropic stress σ'_0 and void ratio e_0 . The maximum friction angle and the dilatancy at peak increase with reduction of the stress and reduction of the initial void ratio.

Simple shear test calculations with the same initial void ratios, materials and vertical stresses σ'_y as in the triaxial tests were made, e.g. Fig. 10. The lateral stresses (σ'_x and σ'_z) have been calculated with $(1 - \sin \varphi_c)\sigma'_y$. The maximum shear stress τ_f of the simple shear test can be transformed to a slope inclination β with the help of (14). This β is plotted in Figs. 11, 12 and 13 together with φ and ψ from the corresponding triaxial calculations for the same material, initial stress and initial void ratio.

The solutions of the calculations with Matsuoka-Nakai are added in these figures. The two approaches show similar results. However, the results of the simple shear calculations with Hypoplasticity agree slightly better with those of the kinematic approach in the case of $\psi = 0$.

7. Triaxial test versus simple and direct shear test

The maximum shear stress τ_f measured in a direct shear test is used to determine the friction angle

$$\tan \phi = \frac{\tau_f}{\sigma'_f} \quad . \quad (23)$$

However, this value is not generally equal to the friction angle φ determined from a triaxial test [25]. In the following plane strain experiments we denote $\varphi_{ps} := \max \varphi_m$, which may be higher than φ (cf. Fig. 5).

Figure 11: Comparison of the different computations for $\psi = 0$.

Abbildung 11: Vergleich verschiedener Berechnungen mit $\psi = 0$.

Figure 12: Comparison of the different computations for $\psi = \varphi/4$.

Abbildung 12: Vergleich verschiedener Berechnungen mit $\psi = \varphi/4$.

Material	φ_c	h_s	n	e_{d0}	e_{c0}	e_{i0}	α	β
Hostun sand	31°	1000 MPa	0.29	0.61	0.91	1.09	0.13	1.00
Hochstetten sand	33°	1500 MPa	0.28	0.55	0.95	1.05	0.25	1.50
Hochstetten gravel	36°	32 000 MPa	0.18	0.26	0.45	0.50	0.1	1.80

Table 1: Material paramters for the hypoplastic constitutive model in version [24]; This hypoplastic model has eight parameters: the critical friction angle φ_c , the granular hardness h_s , the void ratios e_{i0} , e_0 and e_{d0} , the exponents n , α and β .

Tabelle 1: Materialparamter für das hypoplastische Materialmodell in der Verison von [24]; Dieses hypoplastische Modell besitzt acht Parameter: den kritischen Reibungswinkel φ_c , die Granulathärte h_s , die Porenzahlen e_{i0} , e_0 and e_{d0} , die Exponenten n , α und β .

Figure 13: Comparison of the different computations for $\psi = \varphi/3$.

Abbildung 13: Vergleich verschiedener Berechnungen mit $\psi = \varphi/3$.

Vardoulakis and Goldscheider [26] concluded from biaxial tests, that the Coulomb friction angle ϕ in the shear band at peak strength is approximately equal to the maximum of the mobilized plane strain Mohr-Coulomb friction angle φ_{ps} , which is larger than the Mohr-Coulomb friction angle φ in triaxial tests due to the plane strain conditions in biaxial tests. Simple shear experiments of [27] showed that the plane strain Mohr-Coulomb friction angle is higher than the Coulomb friction angle. This is confirmed numerically with discrete elements for simple shear [10] and with Hypoplasticity [28]. Numerical direct shear test with discrete elements [29] also reveals a plane strain Mohr-Coulomb friction angle higher than the Coulomb friction angle. The mechanics discussed above for the infinite slope can be applied if we assume a simple shear deformation in the shear gap. The relation between these two friction angles is in analogy to (4)

$$\tan \phi = \frac{\sin \varphi \cos \psi}{1 - \sin \varphi \sin \psi} \quad (24)$$

for a Mohr-Coulomb failure criterion, i.e. $\phi = \beta$.

The boundary conditions of a direct shear test are equal to the boundary conditions of an infinite slope. The relation $\phi = \beta$ holds irrespective of the material model used for a transformation of ϕ to the material parameter of that model. It seems therefore more appropriate to use direct shear tests to determine a calculation parameter ϕ for a limit state analysis than the triaxial test, since ϕ includes the effect of the dilatancy. This calculated parameter ϕ plays the same role as the reduced friction angle φ^* from (7) [12]. However, the grain size limitation has to be noted. The shear gap should be $(10 \dots 20) \cdot d_{50}$ [30], where d_{50} is the diameter corresponding to 50 % finer in the particle-size distribution. Other effects concerning the construction are discussed in [31, 32].

8. Impact on limit state analyses

Theoretically, reduced strength parameters (7) and (8) should be used in any limit state analysis based on slip lines in soil, i.e. also in the common Bishop's method [33] or rigid body calculations [34]. However, based on our investigations we recommend:

1. When ever possible, the friction angle ϕ derived in a direct shear test in plane strain calculations to be used

$$\varphi^* = \phi = \arctan \frac{\tau_f}{\sigma} \quad (25)$$

One has to be aware on the limitations of such tests, e.g. the maximum grain size of the soil sample.

2. When a friction angle φ is derived from a triaxial test

$$\varphi^* = \min \left\{ \arctan \frac{\sin(1.085\varphi) \cos \psi}{1 - \sin \varphi \sin \psi}, \varphi \right\} \quad (26)$$

can be used if plane strain conditions are assured. That requires the determination of the angle of dilatancy ψ which is not standard in all laboratories. Without knowledge of ψ the approximation $\varphi^* \approx \varphi$ may be used for $\varphi < 35^\circ$.

3. If plane strain conditions are not or only partly fulfilled one should provide a conservative estimate with

$$\varphi^* = \arctan \frac{\cos \psi \sin \varphi}{1 - \sin \psi \sin \varphi} \quad (27)$$

compare (4) of [13, 12].

9. Conclusion

The use of reduced shear parameters has been proposed by several authors for limit state analysis based on slip lines. However, the increased soil strength in plane strain condition is not considered by using these parameters, since they are based on a Mohr-Coulomb failure criterion. The calculations in this article suggest, that the plane strain condition somehow counterbalance the effect of the non-associated flow rule, at least for moderate friction angles less than 35 degrees and moderate stress levels which allow for a dilatancy angle at peak larger than $\varphi/3$. In such cases the strength parameters determined in a standard triaxial test can be used directly in the common limit state calculation methods of geotechnical engineering without taking the dilatancy into account. This verifies the common practice in geotechnical engineering. In particular, the classical limit state relation for infinite slopes $\beta = \varphi$ holds approximately.

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A. Simple Shear with Mohr-Coulomb

The Mohr-Coulomb failure criterion is

$$f = \frac{\sigma'_I + \sigma'_{III}}{2} \sin \varphi + \frac{\sigma'_I - \sigma'_{III}}{2} \quad . \quad (28)$$

For plain strain conditions the principal stresses are

$$\sigma'_{I,III} = \frac{\sigma'_x + \sigma'_y}{2} \pm \sqrt{\frac{(\sigma'_x - \sigma'_y)^2}{4} + \tau_{xy}^2} = \sigma^* \pm \tau^* \quad , \quad (29)$$

from replacing (29) in (28) it follows

$$f = \sigma^* \sin \varphi + \tau^* = \frac{\sigma'_x + \sigma'_y}{2} \sin \varphi + \sqrt{\frac{(\sigma'_x - \sigma'_y)^2}{4} + \tau_{xy}^2} \quad . \quad (30)$$

The plastic potential g for Mohr-Coulomb is defined as

$$g = \sigma^* \sin \psi + \tau^* \quad . \quad (31)$$

For the calculation in the elastic region also the elastic material tensor $\mathbf{C}^e$ is required, which is

$$\mathbf{C}^e = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 & \nu \\ \nu & 1-\nu & 0 & \nu \\ 0 & 0 & 0.5-\nu & 0 \\ \nu & \nu & 0 & 1-\nu \end{bmatrix} \quad . \quad (32)$$

After the yield surface is reached the elastoplastic material tensor $\mathbf{C}^{ep}$ has to be used, which depends on the actual stress state and is calculated with

$$\mathbf{C}^{ep} = \mathbf{C}^e - \frac{\mathbf{C}^e \mathbf{m} \mathbf{n}^\top \mathbf{C}^e}{\mathbf{n}^\top \mathbf{C}^e \mathbf{m}} \quad . \quad (33)$$

Herein, $\mathbf{n}$ and $\mathbf{m}$ are the normals on the yield surface f and the plastic potential g respectively. For the Mohr-Coulomb failure criterion and plane strain $\mathbf{n}$ is

$$\mathbf{n} = \frac{\partial f}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \frac{\partial f}{\partial \sigma'_x} \\ \frac{\partial f}{\partial \sigma'_y} \\ \frac{\partial f}{\partial \tau_{xy}} \\ \frac{\partial f}{\partial \sigma'_z} \end{bmatrix} = \begin{bmatrix} \frac{\sigma'_x - \sigma'_y}{2\tau^*} + \sin \varphi \\ -\frac{\sigma'_x - \sigma'_y}{2\tau^*} + \sin \varphi \\ \frac{2\tau_{xy}}{\tau^*} \\ 0 \end{bmatrix} \quad . \quad (34)$$

The normal on the plastic potential is determined as

$$\mathbf{m} = \frac{\partial g}{\partial \boldsymbol{\sigma}} = \begin{bmatrix} \frac{\partial g}{\partial \sigma'_x} \\ \frac{\partial g}{\partial \sigma'_y} \\ \frac{\partial g}{\partial \tau_{xy}} \\ \frac{\partial g}{\partial \sigma'_z} \end{bmatrix} = \begin{bmatrix} \frac{\sigma'_x - \sigma'_y}{2\tau^*} + \sin \psi \\ -\frac{\sigma'_x - \sigma'_y}{2\tau^*} + \sin \psi \\ \frac{2\tau_{xy}}{\tau^*} \\ 0 \end{bmatrix} \quad . \quad (35)$$

In the equations (34) and (35) the expression $\frac{\sigma'_x - \sigma'_y}{2\tau^*}$ can be substituted by $\cos 2\alpha$ and $\frac{\tau_{xy}}{\tau^*}$ by $\sin 2\alpha$ (see Fig. 14)

For the calculation of the maximum stress, a strain-incremented simple shear test is computed. The shearing proceeds in the xy -plane, with vertical loading in y -direction and

Figure 14: Mohr's circle.

Abbildung 14: Mohr'scher Kreis.

constrained displacement in z-direction. The boundary conditions are hence as follows: zero normal strains in x- and z-directions and a constant normal stress in y-direction. These conditions result in the following system of equations after the yield surface is reached

$$\begin{bmatrix} \dot{\sigma}'_x = ? \\ \dot{\sigma}'_y = 0 \\ \dot{\tau}_{xy} = ? \\ \dot{\sigma}'_z = ? \end{bmatrix} = \mathbf{C}^{ep} \begin{bmatrix} \dot{\epsilon}_x = 0 \\ \dot{\epsilon}_y = ? \\ \dot{\gamma}_{xy} \\ \dot{\epsilon}_z = 0 \end{bmatrix} . \quad (36)$$

The unknown in these equation can be calculated with

$$\dot{\epsilon}_y = -\frac{C_{2,3}^{ep}}{C_{2,2}^{ep}} \dot{\gamma}_{xy} , \quad (37)$$

$$\dot{\sigma}'_x = \left(-\frac{C_{2,3}^{ep} C_{1,2}^{ep}}{C_{2,2}^{ep}} + C_{1,3}^{ep} \right) \dot{\gamma}_{xy} , \quad (38)$$

$$\dot{\tau}_{xy} = \left(-\frac{C_{2,3}^{ep} C_{3,2}^{ep}}{C_{2,2}^{ep}} + C_{3,3}^{ep} \right) \dot{\gamma}_{xy} , \quad (39)$$

$$\dot{\sigma}'_z = \left(-\frac{C_{2,3}^{ep} C_{4,2}^{ep}}{C_{2,2}^{ep}} + C_{4,3}^{ep} \right) \dot{\gamma}_{xy} . \quad (40)$$

The final stress state is reached, when all stress rates are zero. From (38), (39) and (40) it can be seen that this is the case if the terms in brackets get zero. Some manipulation yields for (38) and (40)

$$\sin 2\alpha (\cos 2\alpha + \sin \psi) = 0 . \quad (41)$$

The normal stress rates $\dot{\sigma}'_x$ and $\dot{\sigma}'_z$ vanish for $\alpha = 0^\circ$ ($\sin 2\alpha = 0$) and for $\alpha = \psi/2 + 45^\circ$ ($\cos 2\alpha + \sin \psi = 0$). The term in brackets of (39) is

$$\cos^2 2\alpha + \cos 2\alpha (\sin \varphi + \sin \psi) + \sin \varphi \sin \psi = 0 . \quad (42)$$

The result of the quadratic equation is $\alpha = \psi/2 + 45^\circ$ and $\alpha = \varphi/2 + 45^\circ$.

It can be seen that for the case of $\alpha = \psi/2 + 45^\circ$ all stress increments become zero and also the final stress state is reached.

Figure 15: Mohr's diagram.

Abbildung 15: Mohr'scher Kreis.

With the knowledge of this angle and the normal stress σ_y the associated final shear stress can be calculated as (cf. Fig. 15),

$$\tau^* = \sigma^* \sin \varphi , \quad (43)$$

$$\tau_{xy} = \tau^* \sin 2\alpha , \quad (44)$$

$$\sigma'_y = \tau^* \cos 2\alpha + \sigma^* . \quad (45)$$

With the use of (43) in (45) and (44) the relation of τ_{xy} to σ_y is

$$\frac{\tau_{xy}}{\sigma'_y} = \frac{\sin \varphi \sin 2\alpha}{1 + \sin \varphi \cos 2\alpha} . \quad (46)$$

The equation (4) results by substituting 2α with $\psi + 90^\circ$.

B. Overconsolidation ($K_0 > 1$)

Numerical simple shear experiments with discrete elements [10] showed for $K = 2$ that the angle of non-coaxiality (compare appendix C) starts with $\iota > 0$ and then strongly decreases to $\iota = 8^\circ$ at the shear stress peak of the simple shear experiment, which would yield to a slightly higher slope angel due to (5), than those calculated with a coaxial flow rule. However, the peak shear stress in the simple shear test appears to be approximately the same for $K = 2$ and $K = 1$, where coaxiality $\iota = 0$ holds approximately throughout the entire shear deformation, which is a consequence of the different angle of dilatancy ψ in both situations. Hence, the values for β computed with a coaxial flow rule for $K > 1$ seems to be an acceptable approximation.

It is also in elasto-plastic calculations possible to gain a higher ϕ for an overconsolidated soil. Therefore, it is necessary for the lateral stress at the start to be higher than at the end of a normally consolidated calculation. The lateral stress σ_x is in this case

$$\sigma'_x = \sigma^* - \tau^* \cos 2\alpha . \quad (47)$$

Figure 16: Stress-strain curves for different values of K ($\varphi = 30^\circ$ and $\psi = \varphi/4$).

Abbildung 16: Spannungs-Dehnungslinien für unterschiedliche Werte von K ($\varphi = 30^\circ$ und $\psi = \varphi/4$).

The relation of σ'_x and σ'_y can be calculated with $\alpha = 45^\circ + \psi/2$ and is

$$K = \frac{\sigma'_x}{\sigma'_y} \geq \frac{\sigma^* - \tau^* \cos 2\alpha}{\sigma^* + \tau^* \cos 2\alpha} = \frac{1 + \sin \varphi \sin \psi}{1 - \sin \varphi \sin \psi} \quad (48)$$

If the lateral stress is also higher than in the second solution of (42) ($\alpha = \varphi/2 + 45^\circ$) the shear strain reaches a maximum for the given friction angle. The lateral stress coefficient has to be higher than

$$K = \frac{\sigma'_x}{\sigma'_y} \geq \frac{\sigma^* - \tau^* \cos 2\alpha}{\sigma^* + \tau^* \cos 2\alpha} = \frac{1 + \sin^2 \varphi}{\cos^2 \varphi} \quad (49)$$

and the relation between shear and normal stress is $\tan \varphi$, if (46) is used, what also means $\varphi = \phi$.

In Figure 16 stress-strain-curves for the three different lateral stress coefficient K are plotted. It can be seen that for a normal consolidated soil (blue line, $K_0 = 1 - \sin \varphi$) the shear stress increases until the maximum is reached. For a coefficient higher than in (48) (red line, $K > 1.13$ for this pair of φ and ψ) a peak can be reached, which is higher than in a normal consolidated soil. After the peak the shear stress decreases again until the value of the normal consolidated soil is reached. In the case of a lateral stress coefficient also higher than (49) (in this case $K > 1.67$) a peak value of $\tan \varphi$ can be reached (green line). After that the shear stress also decreases to the normal consolidated value.

Note, that simple shear computations with hypoplasticity do not confirm an increasing ϕ with increasing K , Fig. 17. Moreover, for dense soil the opposite trend is predicted, Fig. 18. Hence, we do not recommend to use the higher ϕ predicted for

Figure 17: Stress-strain curves for different values of K and loose Hostun sand ($\sigma_y = -200$ kPa, $e_0 = 0.822$ kPa).

Abbildung 17: Spannungs-Dehnungslinien für unterschiedliche Werte von K und lockeren Hostun Sand ($\sigma_y = -200$ kPa, $e_0 = 0.822$ kPa).

$K > 1$ by calculations with the Mohr-Coulomb elasto-plastic model in slope stability design.

C. Coaxiality

The angle between the direction of the largest principal stress and the horizontal is defined as α_σ . Analogically the angle between the horizontal axes and the principal strain rate $\dot{\epsilon}_1$ is defined as α_ϵ for the elastoplastic calculations and α_D for the hypoplastic calculations.

For the elastoplastic calculation it can be seen in Fig. 19 that the stress and strain rate are not coaxial at the beginning. However, the directions converge with on going shearing and are practically the same after a long shearing. As already mentioned in section 3, α_σ is for $K < 1$ always smaller than α_ϵ .

For the hypoplastic calculation the coaxiality of stress and deformation rate tensor is also not given (cf. Fig. 20) at the beginning. Just as in the elasto-plastic calculations after a long shearing the stress tensor and deformation rate tensor are coaxial.

Figure 18: Stress-strain curves for different values of K and dense Hostun sand ($\sigma_y = -100$ kPa, $e_0 = 0.627$ kPa).

Abbildung 18: Spannungs-Dehnungslinien für unterschiedliche Werte von K und dichten Hostun Sand ($\sigma_y = -100$ kPa, $e_0 = 0.627$ kPa)

Figure 19: The principal direction of the stress tensor α_σ and the strain rate tensor α_ε in an elasto-plastic calculation ($\varphi = 30^\circ$, $\psi = \varphi/4$).

Abbildung 19: Die Hauptspannungsrichtung α_σ und die Hauptrichtung des Verzerrungsrate tensors α_ε in einer elastoplastischen Berechnung ($\varphi = 30^\circ$, $\psi = \varphi/4$).

Figure 20: The principal direction of the stress tensor α_σ and the deformation tensor α_D in a hypoplastic calculation ($\sigma_x = \sigma_z = 4.85$ kPa, $\sigma_y = 10$ kPa, $e = 0.667$, Hostun Sand).

Abbildung 20: Die Hauptspannungsrichtung α_σ und die Hauptrichtung des Deformationsrate tensors α_D in einer hypoplastischen Berechnung ($\sigma_x = \sigma_z = 4.85$ kPa, $\sigma_y = 10$ kPa, $e = 0.667$, Hostun Sand).